A THEORETICAL APPROACH ON THE SCALING OF THE SECOND-ORDER CROSS STRUCTURE FUNCTION IN HOMOGENEOUS SHEAR TURBULENCE

Le Fang
Ecole Centrale de Pékin
Beihang University
100191 Beijing, P.R. China
email: le.fang@buaa.edu.cn

Li-Po Wang
UM-SJTU Joint Institute
Shanghai Jiao Tong University
200240 Shanghai, P.R. China
email: lipo.wang@sjtu.edu.cn

Introduction

The assumption of *isotropic* turbulence is a quite rigorous requirement in most of the practical applications. By contrast, the concept of "*local isotropy*", which was introduced by Kolmogorov (1941), only assumes that the small-scale turbulence is isotropic and is more realistic. Large-eddy simulation (LES) is one of the applications of this assumption, as *local isotropy* is the basis of most subgrid models (c.f. Smagorinsky (1963); Métais & Lesieur (1992); Cui *et al.* (2004); Fang *et al.* (2009)). In these models the small scales, *i.e.* the subgrid scales (SGS), are assumed to be *local isotropic*, so that the homogeneous isotropic theories can be applied. However, limitations have been found in the simulation of shear turbulence, so some recent anisotropic models have to avoid assuming the *local isotropy* (c.f. Lévêque *et al.* (2007); Cui *et al.* (2007)). Till now, the effect of *local anisotropy* in the existing SGS models has not been carefully studied. In order to better develop these SGS models, it is necessary to investigate the properties of *local anisotropy* affected by the mean shear, which is one of the simplest anisotropic conditions.

There are already many spectral researches on the *local anisotropy* in shear turbulence, which pointed out that the mean shear causes a turning effect in spectral space (c.f. Phillips (1969); Saddoughi & Veeravalli (1994); He & Zhang (2006)). However these spectral conclusions are difficult to be used in practical LES, while a study in physical space is required. In physical space, we can define the structure functions which represent the *local* properties as following:

$$D_{i_1 i_2 \dots i_m}(\boldsymbol{r}) = \left\langle \delta u'_{i_m}(\boldsymbol{r}) \delta u'_{i_m}(\boldsymbol{r}) \dots \delta u'_{i_m}(\boldsymbol{r}) \right\rangle, \quad (1)$$

in which $\boldsymbol{u}$ is the velocity, $\langle\rangle$ means taking ensemble average, $\bullet'$ is the fluctuating part, and $\delta\boldsymbol{u}(\boldsymbol{r}) = \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{u}(\boldsymbol{x})$ is the velocity increment at two-point distance $\boldsymbol{r}$. In particular we are intrested in the second-order cross structure function

$$D_{12}(r\boldsymbol{e}_1) = \left\langle \delta u'_1(r\boldsymbol{e}_1) \delta u'_2(r\boldsymbol{e}_1) \right\rangle, \quad (2)$$

where x_1 is the streamwise direction and x_2 is the normal direction, *i.e.* the mean velocity is $\langle \boldsymbol{u} \rangle = \gamma x_2 \boldsymbol{e}_1$ with γ the mean shear. Similar to the spectral correlation function $\hat{R}_{12}$ in spectral space, this term is the most important cross term in physical space which represents the *local anisotropy*. From the assmption of *local isotropy* there should be $D_{12}(r\boldsymbol{e}_1) \equiv 0$, but the experiments of Kurien & Sreenivasan (2000) showed non-zero results. The measurements were done in an atmospheric boundary layer, showing that the cross term $D_{12}(r\boldsymbol{e}_1) = 0$ satisfying the r^2 scaling in dissipative range and $r^{1.12}$ (or $r^{1.22}$) scaling in inertial range. However till now, there is no theoretical explanation for these scalings. Existed works usually introduce the group theory and expand the structure functions in spherical harmonics (for example Casciola *et al.* (2007)) but did not provide any analytical scaling.

This scaling is also important in the LES modeling of shear turbulence. From Cui *et al.* (2007) $D_{12}(r\boldsymbol{e}_1)$ acts as a source term of the filtered Kolmogorov equation, and explicitly affects the subgrid viscosity. One of our SGS models Fang *et al.* (2009) also showed that $D_{12}(r\boldsymbol{e}_1)$ could be directly related to subgrid stress. Therefore, in order to better perform the SGS models in shear turbulence, we should investigate the behavoir of the cross structure function $D_{12}(r\boldsymbol{e}_1)$, especially in inertial range where LES filters always locate.

In this letter, we extend our previous works, for example Cui *et al.* (2007); Fang *et al.* (2010, 2009, 2014); Yao *et al.* (2014), to homogeneous shear turbulence, and obtain a scaling law of the cross structure function $D_{12}(r\boldsymbol{e}_1)$ based on some additional assumptions. For simplification, the effect of filter has not been considered in this letter, however it will be further investigated by using the similar method as in Ma *et al.* (2011) and Fang *et al.* (2014). The scaling exponent shows extremely good agreement with experimental results and encourages further researches.

In homogeneous shear turbulence, we can decompose the velocity into mean and fluctuating parts:

$$u_i = \langle u_i \rangle + u_i' = \gamma x_2 \delta_{i1} + u_i', \tag{3}$$

and the governing equation for the fluctuating part reads

$$\frac{\partial u_i'}{\partial t} + \gamma x_2 \frac{\partial u_i'}{\partial x_1} + u_k' \frac{\partial u_i'}{\partial x_k} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \nu \frac{\partial^2 u_i'}{\partial x_k \partial x_k}. \tag{4}$$

Following the same progress as Cui *et al.* (2007), we can write Eq. (4) at another point $\boldsymbol{x}^* = \boldsymbol{x} + \boldsymbol{r}$, and obtain the governing equation for velocity increment $\delta \boldsymbol{u}'$. Multiplying the equations of $\delta u_i'$ and $\delta u_j'$ and taking ensemble average, we have the following result:

$$\begin{aligned} &\frac{\partial D_{ij}(\boldsymbol{r})}{\partial t} + \frac{\partial D_{ijk}(\boldsymbol{r})}{\partial r_k} + \gamma r_2 \frac{\partial D_{ij}(\boldsymbol{r})}{\partial r_1} \\ &+ \gamma\left(\delta_{i1} D_{j2}(\boldsymbol{r}) + \delta_{j1} D_{i2}(\boldsymbol{r})\right) \\ &= \frac{2}{\rho}\langle \delta p'(\boldsymbol{r}) \delta S_{ij}'(\boldsymbol{r})\rangle + 2\nu \frac{\partial^2 D_{ij}(\boldsymbol{r})}{\partial r_k \partial r_k} - 4\nu \left\langle \frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}\right\rangle, \end{aligned} \tag{5}$$

where $S_{ij}' = \frac{1}{2}\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)$ is fluctuating shear rate.

In order to simplify the investigation, we have the following assumptions:

1.
$$\frac{\partial D_{ij}(\boldsymbol{r})}{\partial t} = 0. \tag{6}$$

This assumes the steadiness of the small scale turbulence, as many existed works did, for example see Cui *et al.* (2004); Fang *et al.* (2009); Cui *et al.* (2007).

2.
$$D_{ijr}(\boldsymbol{r}) = S D_{ij}(r)^{3/2}, \tag{7}$$

where the subscript r means the direction of $\boldsymbol{r}$. S is assumed as constant skewness which is always negative, thus

$$\frac{\partial D_{ijk}(\boldsymbol{r})}{\partial r_k} = S\frac{dD_{ij}(r)^{3/2}}{dr} + \frac{2SD_{ij}(r)^{3/2}}{r}. \tag{8}$$

This assumption of closure is similar to the Extended Scale Similarity (ESS) theory (Benzi *et al.* (1995)), and has been shown to be reasonable in both dissipative and inertial ranges (see Fang *et al.* (2010)). Althouth not correct in the transition range, it does not affect the investigation of this letter since we only focus on the scaling laws in the dissipative range and inertial range.

3.
$$T_{12} = \frac{2}{\rho}\langle \delta p'(r\boldsymbol{e}_1) \delta S_{12}'(r\boldsymbol{e}_1)\rangle = 0. \tag{9}$$

This correlation between the increment of pressure and the increment of fluctuating shear rate has been studied by many works. Hill (1997) obtained $T_{ij} = 0$ in the case of *local isotropy*. Later, Alvelius & Johansson (2000) showed that T_{ij} is almost zero in the range $r \ll L$ with L energy-containing scale, from a numerical simulation of anisotropic turbulence. In this letter we do not consider the scaling in energy-containing range, thus this assumption could be reasonable.

From these assumptions, we write the governing equation for $D_{12}(r\boldsymbol{e}_1)$ from Eq. (5):

$$\begin{aligned} &S\frac{dD_{12}(r\boldsymbol{e}_1)^{3/2}}{dr} + \frac{2SD_{12}(r\boldsymbol{e}_1)^{3/2}}{r} + \gamma D_{22}(r\boldsymbol{e}_1) \\ &= \frac{2\nu}{r^2}\frac{d}{dr}\left(r^2 \frac{dD_{12}(r\boldsymbol{e}_1)}{dr}\right) - 4\varepsilon_{12}, \end{aligned} \tag{10}$$

where $\varepsilon_{12} = \nu\left\langle \frac{\partial u_1'}{\partial x_k}\frac{\partial u_2'}{\partial x_k}\right\rangle$. From the experiment of Saddoughi & Veeravalli (1994) we have approximately $D_{22}(r\boldsymbol{e}_1) = D_{nn}(r)$, where D_{nn} is the second-order transverse structure function in homogeneous isotropic turbulence. The terms in the right-hand side of Eq. 10 are related to the molecular viscosity. We can then define a non-dimensional parameter $\beta = \frac{\nu}{\gamma\eta^2}$ to denote the proportion between the viscous terms and the shear term, with η the Kolmogorov scale. In the following parts, we first introduce the analytical solution of negligible viscous terms comparing with the shear effect, *i.e.* $\beta = 0$; then numerically consider the effect of viscous terms when $\beta \neq 0$.

Neglecting the viscous terms under extremely strong shear

When the viscous effect is negligible when comparing with the shear effect, Eq. (10) becomes:

$$S\frac{dD_{12}(r\boldsymbol{e}_1)^{3/2}}{dr} + \frac{2SD_{12}(r\boldsymbol{e}_1)^{3/2}}{r} + \gamma D_{nn}(r) = 0. \tag{11}$$

The solution of $D_{12}(r\boldsymbol{e}_1)$ is (note the initial condition $D_{12}(\boldsymbol{0}) = 0$)

$$D_{12}(r\boldsymbol{e}_1) = \left(-\gamma\frac{\int_0^r r^2 D_{nn}(r)\,dr}{Sr^2}\right)^{2/3}. \tag{12}$$

In dissipative range $D_{nn}(r) \propto r^2$, and we obtain

$$D_{12}(r\boldsymbol{e}_1) \propto \gamma^{2/3} r^2. \tag{13}$$

In inertial range $D_{nn}(r) \propto r^{2/3}$, thus

$$D_{12}(r\boldsymbol{e}_1) \propto \gamma^{2/3} r^{10/9} \approx \gamma^{2/3} r^{1.111}. \tag{14}$$

If we consider the anomalous scaling that $D_{nn}(r) \propto r^{0.69}$ (c.f. She & Leveque (1994)) in inertial range instead, this scaling changes to about $r^{1.127}$. For simplification, in this letter we do not consider this anomalous effect. From the experiments of Kurien & Sreenivasan (2000), the r^2 law in dissipative range is well satisfied. In inertial range, it was found that $D_{12}(r\boldsymbol{e}_1) \propto r^{1.22}$ at 0.54m (the distance between the measure point and the ground) and $D_{12}(r\boldsymbol{e}_1) \propto r^{1.12}$ at 0.27m. They are both in agreement with Eq. (14). Note that we have assumed strong shear effect comparing with viscous effect, while the shear effect at 0.27m is more strong than that at 0.54m, and the result $r^{1.12}$ is much more close to our theoretical value $r^{1.111}$. This perfect agreement proves the correctness of our theoretical approach. In addition, the denominator 9 of the scaling exponent $10/9$ might be related with the dimensional analysis of Bos & Bertoglio (2007) in spectral space, where a scaling of $k^{-23/9}$ was found for a cross term between velocity and passive scalar.

Considering the viscous effect under moderate shear

It is difficult to obtain an analytical solution for Eq. (10) when $\beta \neq 0$. Instead, it is numerically solved in this section. As explained in Fang *et al.* (2010), Batchelor's formula could be appropriate for modeling $D_{nn}(r)$ (Batchelor (1951))

$$D_{ll}(r) = \frac{2u_0^2(\frac{r}{\eta})^{2/3}}{[1+(C_b\frac{\eta}{r})^2]^{2/3}}, D_{nn}(r) = D_{ll}(r) + \frac{1}{2}\frac{dD_{ll}(r)}{dr}, \tag{15}$$

where $D_{ll}(r)$ is the longitude structure function, $u_0 = (\nu\varepsilon)^{1/4}$ is the Kolmogorov velocity, and $C_b = 30^{3/4}$ is a constant. The skewness of Eq. (7) is fixed as $S = -0.38$ as was proposed in Fang *et al.* (2010).

We then calculate Eq. (10) numerically by giving different non-dimensional value β. In order to better analyze the results, the scaling exponent of a structure function is introduced as $n(r) = \frac{dD(r)}{dr}\frac{r}{D(r)}$, where $D(r)$ can be any structure function. The corresponding results are shown in Fig. 1. When $\beta = 0$, the negligible viscous effect leads to the r^2 scaling in dissipative range and $r^{10/9}$ scaling in inertial range, as was explained in the above parts. The r^2 scaling in dissipative range is because of by Taylor expansion, so it can not be affected by different β. Besides, we find that the increasing β does not change the asymptotic value $10/9$ in inertial range. In Fig. 1 it is shown that $\beta < 1$ causes an asymptotic scaling $n \to 10/9$ when $r/\eta > 100$, besides we also observed $n \to 10/9$ at very large r for larger β (they are shown in the subfigure of Fig. 1). When $\beta \neq 0$, the small-scale ($r/\eta < 10$) changes rapidly, which means that the viscous terms mainly affect the small-scale scaling. When r is small, strong viscous effect causes the classical scaling law which varies from 2 to $2/3$, like the lines of $\beta = 100$ and 1000 in Fig. 1; when r is large enough, all the scaling exponents of $D_{12}(r\boldsymbol{e}_1)$ tend to the theoretical value $10/9$. A conclusion can be therefore made that, in homogeneous shear turbulence, the viscous terms determine the small-scale scaling and the shear term determines the large-scale scaling. This conclusion could be reasonable since the dissipation is always a small-scale phenomenon and the mean shear could be regarded as a large-scale flow structure. Note that this "large-scale" does not mean the energy-containing scale, since it is still in inertial range where LES filters always locate. Therefore, the shear effect denotes the *local anisotropy* and should be considered in anisotropic LES modeling.

Conclusion

In this letter we are interested in the theoretical explanation of the scaling of the second-order cross structure function. An approach is applied by simplifying the governing equation of second-order structure function. A non-dimensional parameter β is defined to denote the proportion between the viscous and shear effects. When $\beta = 0$, the viscous terms are negligible, and a $r^{10/9}$ scaling is obtained in inertial range, showing extremely good agreement with existing experimental results; when $\beta \neq 0$, the viscous effect is studied numerically, showing that the viscous terms determine the small-scale scaling and the shear term determines the large-scale scaling. The understanding of this cross structure function shows the property of *local anisotropy* in homogeneous shear turbulence, and might help us to improve anisotropic SGS models. The assumptions (6), (7) and (9) are expected be verified from DNS shear flow and experiments to better support this theory in the future.

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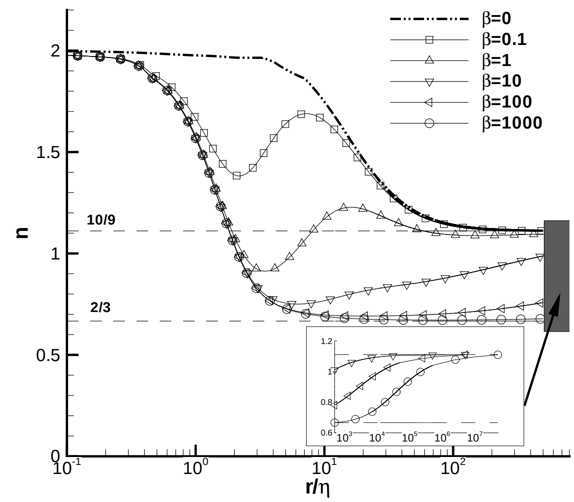

Figure 1. The scaling exponent of $D_{12}(r\boldsymbol{e}_1)$, with different non-dimensional parameter β.

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