TURBULENT ENERGY DENSITY IN SCALE SPACE BASED ON TWO-POINT VELOCITY CORRELATION

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ABSTRACT

The energy spectrum contains information not only on the intensity but also on the length scale of the turbulent fluctuations. Its behavior has been studied in detail for homogeneous isotropic turbulence. On the other hand, one-point statistics such as the turbulent kinetic energy is treated for inhomogeneous turbulence. This is partly because the Fourier transform cannot be performed in inhomogeneous directions. In this work, instead of the energy spectrum in the wavenumber space, the energy density in the scale space is defined using the two-point velocity correlation. Its transport equation for inhomogeneous turbulence is derived. The energy transfer in the scale space is evaluated using the direct numerical simulation of homogeneous isotropic turbulence and turbulent channel flow. The energy density was compared with the energy spectrum in homogeneous isotropic turbulence. The energy transfer from the large to the small scales is observed in both isotropic and inhomogeneous turbulence.

INTRODUCTION

In order to better understand inhomogeneous turbulence, it must be useful to examine the energy transport not only in the physical space but also in the wavenumber space. Instead of the energy spectrum, the second-order velocity structure function $\langle (u_i'(\mathbf{x}+\mathbf{r})-u_i'(\mathbf{x}))^2\rangle$ was treated as the scale energy and its transport in $\mathbf{r}$ space was discussed (Hill, 2002; Marati et al., 2004; Davidson, 2004; Cimarelli et al., 2012). Its transport equation is a natural extension of the Kolmogorov equation for isotropic turbulence to the inhomogeneous case. However, its meaning in the limit of $r \to \infty$ for inhomogeneous turbulence is not clear because it corresponds to the sum of the energies at very distant two points as $\langle u_i'^2(\mathbf{x}+\mathbf{r})\rangle + \langle u_i'^2(\mathbf{x})\rangle$.

In this work, we propose another definition of the scale energy based on the two-point velocity correlation $\langle u_i'(\mathbf{x})u_i'(\mathbf{x}+\mathbf{r})\rangle$. In the limit of $r \to 0$ it corresponds to the turbulent energy at a single point $\mathbf{x}$ unlike the structure function. The velocity correlation can be considered the part of the turbulent energy whose scale is greater than r. We then define the energy density using the gradient of the two-point velocity correlation. Similar energy density in the scale space was proposed using the gradient of the structure function (Davidson, 2004). We expect that the energy density based on the two-point velocity correlation is suitable in discussing the energy transfer in the scale space. We derive the transport equation for the energy density in inhomogeneous turbulence. As a first step we examine the transport equation using the direct numerical simulation (DNS) data of homogeneous isotropic turbulence and turbulent channel flow.

ENERGY DENSITY IN SCALE SPACE

The transport equation for the energy spectrum $E(k)$ for isotropic turbulence is given by

$$\frac{\partial}{\partial t}E(k) = T(k) - \varepsilon(k) + F(k) \tag{1}$$

$$T(k) = 2\pi k^2 \iint d\mathbf{p}\, d\mathbf{q} S(\mathbf{k},\mathbf{p},\mathbf{q}) \tag{2}$$

$$\varepsilon(k) = 2\nu k^2 E(k) \tag{3}$$

where $F(k)$ is the forcing term. The energy flux can be defined as

$$\Pi(k) = \int_k^\infty dk' T(k'), \quad T(k) = -\frac{\partial}{\partial k}\Pi(k) \tag{4}$$

The energy transfer can be discussed partly because the energy spectrum is the energy density in the wavenumber space and satisfies

$$K = \int_0^\infty dk E(k) \tag{5}$$

The energy spectrum can also be considered the energy whose length scale is π / k .

For inhomogeneous turbulence, it is not always possible to perform Fourier transform. Instead, we can treat the second-order structure function or the two-point velocity correlation $Q_{ii}(\mathbf{x},\mathbf{r}) = \langle u_i'(\mathbf{x})u_i'(\mathbf{x}+\mathbf{r})\rangle$ to examine the energy transfer in the space of scale like the

Kármán-Howarth equation. The correlation $Q_{ii}(\mathbf{x},\mathbf{r})/2$ can be considered the energy whose scale is larger than r and it does not exactly represent the energy density in the scale space (r space). Instead of $Q_{ii}(\mathbf{x},\mathbf{r})/2$, we introduce the following quantity as the scale energy:

$$E(r) = -\frac{1}{2}\frac{\partial}{\partial r}Q_{ii}(r) = -\frac{1}{2}\frac{\partial}{\partial r}\langle u_i'(\mathbf{x})u_i'(\mathbf{x}+\mathbf{r})\rangle \qquad (6)$$

Since this energy satisfies

$$K = \int_0^\infty dr E(r) \qquad (7)$$

it represents the energy density in r space. This energy density can be examined even in inhomogeneous turbulence as

$$E(\mathbf{x},\mathbf{r}) = -\frac{1}{2}\frac{\partial}{\partial r}Q_{ii}(\mathbf{x},r,\theta,\phi) \qquad (8)$$

using the one-dimensional integral in the r direction with angles θ and ϕ fixed.

TRANSPORT EQUATION FOR SCALE ENERGY

The transport equation for $E(\mathbf{x},\mathbf{r})$ can be written as

$$\begin{aligned}
&\frac{D}{Dt}E(\mathbf{x},\mathbf{r}) \\
&= \frac{1}{2}\frac{\partial}{\partial r}\langle u_k'(\mathbf{x})u_i'(\mathbf{x}+\mathbf{r})\rangle\left(\frac{\partial}{\partial x_k}U_i(\mathbf{x}) + \frac{\partial}{\partial x_i}U_k(\mathbf{x})\right) \\
&+ \frac{1}{2}\frac{\partial}{\partial r}\left[\langle u_k'(\mathbf{x}+\mathbf{r})u_i'(\mathbf{x})\rangle\frac{\partial}{\partial x_k}(U_i(\mathbf{x}+\mathbf{r}) - U_i(\mathbf{x}))\right] \\
&+ \nu\frac{\partial}{\partial r}\left\langle\frac{\partial}{\partial x_k}u_i'(\mathbf{x})\frac{\partial}{\partial x_k}u_i'(\mathbf{x}+\mathbf{r})\right\rangle \\
&+ \frac{\partial}{\partial x_k}\left(\frac{1}{2}\frac{\partial}{\partial r}\langle u_k'(\mathbf{x})u_i'(\mathbf{x})u_i'(\mathbf{x}+\mathbf{r})\rangle\right) \\
&+ \frac{\partial}{\partial x_i}\left(\frac{1}{2}\frac{\partial}{\partial r}\langle p'(\mathbf{x})u_i'(\mathbf{x}+\mathbf{r}) + p'(\mathbf{x}+\mathbf{r})u_i'(\mathbf{x})\rangle\right) \\
&+ \nu\frac{\partial^2}{\partial x_k \partial x_k}E(\mathbf{x},\mathbf{r}) \\
&+ \frac{1}{2}\frac{\partial}{\partial r}\frac{\partial}{\partial r_k}[(U_k(\mathbf{x}+\mathbf{r}) - U_k(\mathbf{x}))\langle u_i'(\mathbf{x})u_i'(\mathbf{x}+\mathbf{r})\rangle] \\
&\quad + \frac{1}{2}\frac{\partial}{\partial r}\frac{\partial}{\partial r_k}\langle(u_k'(\mathbf{x}+\mathbf{r}) - u_k'(\mathbf{x}))u_i'(\mathbf{x})u_i'(\mathbf{x}+\mathbf{r})\rangle \qquad (9)
\end{aligned}$$

By integrating each term from $r = 0$ to ∞, (9) is reduced to the transport equation for the turbulent energy K. On the right-hand side of (9) the first and second terms correspond to the energy production, the third term to the dissipation, and the fourth to sixth terms to the diffusion in the K equation. The remaining seventh and eighth terms represent the energy transfer in r space.

For homogeneous isotropic turbulence the above transport equation is rewritten as

$$\frac{\partial}{\partial t}E(r) = T_E(r) - \varepsilon_E(r) \qquad (10)$$

$$T_E(r) = \frac{1}{2}\frac{\partial}{\partial r}\frac{\partial}{\partial r_k}\langle(u_k'(\mathbf{x}+\mathbf{r}) - u_k'(\mathbf{x}))u_i'(\mathbf{x})u_i'(\mathbf{x}+\mathbf{r})\rangle \quad (11)$$

$$\varepsilon_E(r) = -\nu\frac{\partial}{\partial r}\left\langle\frac{\partial}{\partial x_k}u_i'(\mathbf{x})\frac{\partial}{\partial x_k}u_i'(\mathbf{x}+\mathbf{r})\right\rangle \qquad (12)$$

If the external force exists, the following term $F_E(r)$ is added to the right-hand side of (10).

ANALYSIS USING ISOTROPIC TURBULENCE DNS

The transport equation given by (9) represents the energy transfer in the physical and scale spaces. It must be interesting but very complicated to examine the energy transfer in both spaces in inhomogeneous turbulence. In this work, we first examine the energy transfer in the scale space for homogeneous isotropic turbulence. We carry out DNS of isotropic turbulence with and without external forcing using 512^3 grid points. We will show results of two runs: decaying turbulence with initial spectrum $E(k) \propto k^4 \exp(-2(k/k_p)^2)$ where $k_p = 3.5$ (Case 1) and steady turbulence with external forcing at $k = 2.5 - 4.5$ (Case 2). The Reynolds number R_λ is 75 and 121 for Cases 1 and 2, respectively.

Figure 1 and 2 show the energy spectra for Cases 1 and 2, respectively. Since the Reynolds number is not very high, the inertial range where $E(k) \propto k^{-5/3}$ is narrow and is located at $|k|$: 20 for Case 1. Figure 3 shows the energy density $E(r)$ for Case 1. The energy corresponding to $E(k) \propto k^{-5/3}$ is $E(r) \propto r^{-1/3}$; its profile is also plotted in Fig. 3. Inertial range is located at $|r|$ 0.1. The Kolmogorov length scale is $\eta = 0.0062$ for this flow. To see the detailed profile at the small scale of r, we plot the spectrum in the semi-log scale in Fig. 4. The energy density is less than $r^{-1/3}$ at $r < 0.07$ in Fig. 4 because of the small energy spectrum at the dissipation range shown in Fig. 1.

Next we compare the energy transfer between the wavenumber and scale spaces. Figure 5 shows terms in the energy transport equation (1) in the wavenumber space for Case 1. The transfer term $T(k)$ is negative in the low wavenumber region and positive in the high wavenumber region, representing the forward energy cascade. At the dissipation range the transfer and dissipation terms are balanced to each other. Figure 6 shows the energy transport equation (1) for Case 2. In the low wavenumber region the large positive value of the forcing term $F(k)$ is balanced by the negative value of the transfer term $T(k)$. The profiles in the high wavenumber region are similar to those for Case 1 plotted in Fig. 5.

Figure 7 shows terms in the energy transport equation (10) in the scale space as functions of r for Case 1. The small (large) scale region in Fig. 7 corresponds to the high (low) wavenumber region in Fig. 5. The transfer term $T_E(r)$ is negative in the large scale region and positive in the small scale region, representing the energy transfer from the large scale to the small scale. Although the amplitude of terms is different between Figs. 5 and 7, the tendency of the energy transfer is the same. Figure 8 shows terms in the energy transport equation (10) in the scale space for Case 2. Unlike Fig. 6, the profile of the forcing term $F_E(r)$ is rather broad in Fig. 8. However, the behavior of the energy transfer is similar to that for Fig. 6. The profiles shown in Figs. 7 and 8 suggest that the energy density defined as (6) can be useful for examining the energy transfer in the scale space.

ANALYSIS USING CHANNEL FLOW DNS

In order to assess whether the energy density is applied to inhomogeneous turbulence, we examine the transport equation using the DNS data of turbulent channel flow. The size of the computational domain is $L_x \times L_y \times L_z = 2\pi \times 2 \times \pi$. The number of grid points is $N_x \times N_y \times N_z = 512 \times 192 \times 512$. The Reynolds number based on the friction velocity u_τ and the channel half width $L_y / 2$ is set to $Re_\tau = 395$. Physical quantities are nondimensionalized using u_τ and $L_y / 2$ The periodic boundary conditions are used in the streamwise and spanwise directions and no-slip conditions are imposed at the wall ($y = \pm 1$). Statistics are obtained by averaging over x-z plane and over a time period of 20.

In this work we define the energy density for channel flow as

$$E(y, r_x) = -\frac{1}{2}\frac{\partial}{\partial r_x} Q_{ii}(y, r_x) \tag{13}$$

Its transport equation is given by

$$\begin{aligned}
&\frac{\partial}{\partial t}E(y, r_x) \\
&= \frac{1}{2}\frac{\partial}{\partial r_x}\langle u'_y(\mathbf{x})u'_x(\mathbf{x}+r_x\mathbf{e}_x) + u'_x(\mathbf{x})u'_y(\mathbf{x}+r_x\mathbf{e}_x)\rangle\rangle \frac{\partial U_x}{\partial y} \\
&+ \nu\frac{\partial}{\partial r}\left\langle \frac{\partial}{\partial x_k}u'_i(\mathbf{x})\frac{\partial}{\partial x_k}u'_i(\mathbf{x}+r_x\mathbf{e}_x)\right\rangle \\
&+ \frac{\partial}{\partial y}\left(\frac{1}{2}\frac{\partial}{\partial r_x}\langle u'_y(\mathbf{x})u'_i(\mathbf{x})u'_i(\mathbf{x}+r_x\mathbf{e}_x)\rangle\right) \\
&+ \frac{\partial}{\partial y}\left(\frac{1}{2}\frac{\partial}{\partial r}\langle p'(\mathbf{x})u'_y(\mathbf{x}+r_x\mathbf{e}_x) + p'(\mathbf{x}+r_x\mathbf{e}_x)u'_y(\mathbf{x})\rangle\right) \\
&+ \nu\frac{\partial^2}{\partial y^2}E(y, r_x) \\
&+ \frac{1}{2}\frac{\partial}{\partial r_x}\frac{\partial}{\partial r_k}\langle(u'_k(\mathbf{x}+r_x\mathbf{e}_x) - u'_k(\mathbf{x}))u'_i(\mathbf{x})u'_i(\mathbf{x}+r_x\mathbf{e}_x)\rangle \quad (14)
\end{aligned}$$

where $\mathbf{e}_x$ is the unit vector in the x direction. The right-hand side consists of the production, dissipation, turbulent diffusion, pressure diffusion, viscous diffusion, and transfer terms.

Figure 9 shows the profile of the energy density as a function of r_x for $y = -0.81$ ($y^+ = 76$) in the log layer. The prolife is similar to that for homogeneous isotropic turbulence shown in Fig. 3. Figure 10 shows the profiles of terms in the transport equation for the energy density given by (14) as functions of r_x for $y = -0.81$ ($y^+ = 76$). The production term shows positive values at relatively broad region. The energy transfer term is negative at large scales and positive at small scales, representing the energy flux from the large to the small scales. The dissipation term is negative at small scales and is balanced by the transfer term. In Figs. 8 and 10, the behavior of the transfer and dissipation terms is very similar between the homogeneous isotropic turbulence and turbulent channel flow although the region of the forcing and production terms are different.

CONCLUSIONS

The energy density in the scale space was introduced on the basis of the two-point velocity correlations. The transport equation for the energy density was derived for inhomogeneous turbulence. The energy transfer in the scale space was evaluated using the DNS of homogeneous isotropic turbulence and turbulent channel flow. The energy transfer from the large to the small scales is observed in both flows. It was shown that the energy density is useful for examining the energy transfer in the scale space.

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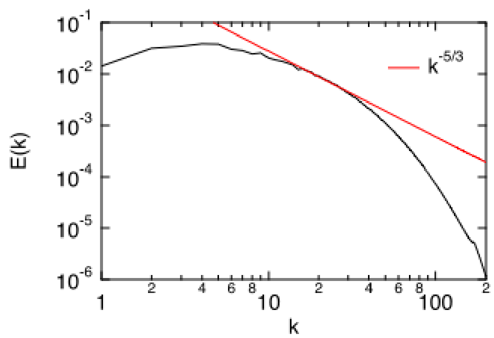

Figure 1. Profile of energy spectrum $E(k)$ for Case 1.

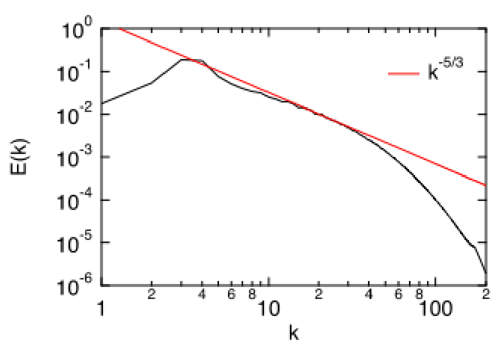

Figure 2. Profile of energy spectrum $E(k)$ for Case 2.

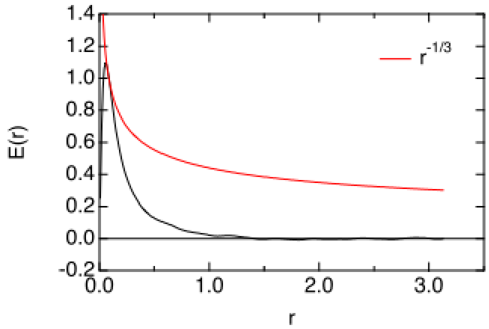

Figure 3. Profile of energy density $E(r)$ for Case 1.

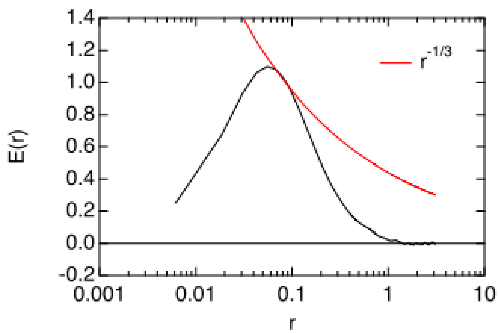

Figure 4. Profile of energy density $E(r)$ for Case 1. (semi-log scale)

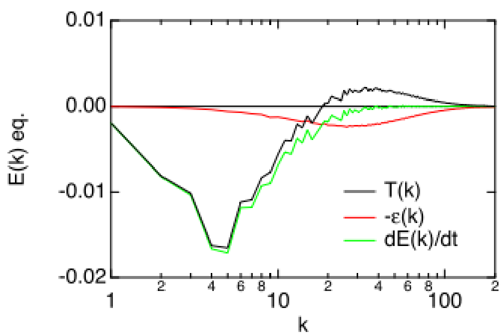

Figure 5. Profiles of transport equation (1) for $E(k)$ for Case 1.

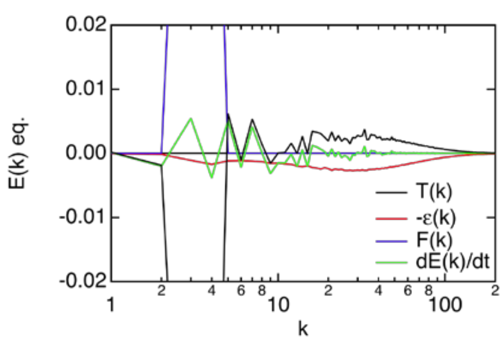

Figure 6. Profiles of transport equation (1) for $E(k)$ for Case 2.

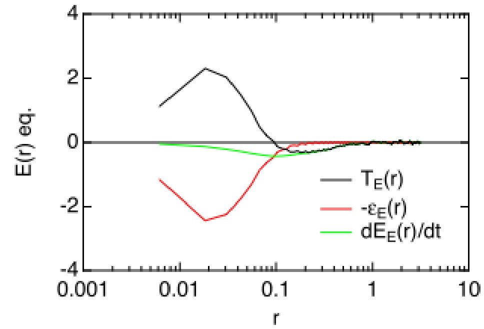

Figure 7. Profiles of transport equation (10) for $E(r)$ for Case 1.

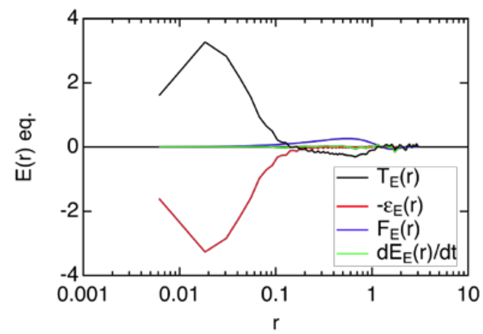

Figure 8. Profiles of transport equation (10) for $E(r)$ for Case 2.

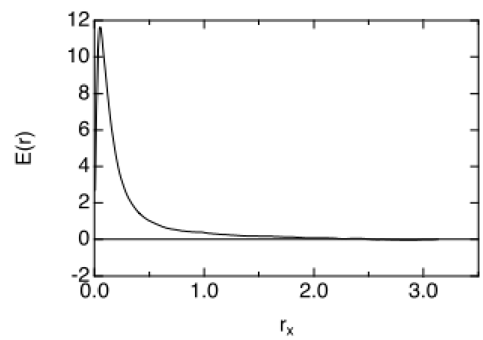

Figure 9. Profile of energy density $E(y, r_x)$ as a function of r_x for $y^+ = 76$ for channel flow.

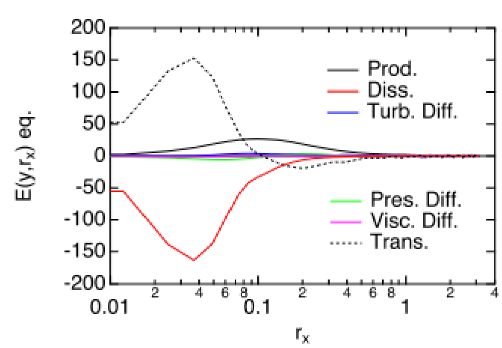

Figure 10. Profile of transport equation (14) as a function of r_x for $y^+ = 76$ for channel flow.