

# Turbulent velocity fluctuations need not be Gaussian

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It is noted that the central limit theorem does not constrain the probability distribution of the components of the turbulent velocity fluctuations to be Gaussian, if their spectral slope is steeper than  $k^{-1}$ . It is shown that, in a model of homogeneous turbulence in which each wavenumber of the energy spectrum predominantly receives contributions from eddies with roughly homogeneous amplitudes and coherence lengths, the p.d.f. would be slightly sub-Gaussian. This agrees with the available experimental evidence.

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## 1. Introduction

Although there is a large literature on the non-Gaussian nature of the probability density functions of the turbulent velocity gradients, recently reviewed by Sreenivasan & Antonia (1997), the p.d.f. of the velocity fluctuations in homogeneous turbulence is often assumed to be Gaussian. Published experimental results show that this is almost true, although few of them are complete enough to really check the assumption. Those who do, tend to find slightly sub-Gaussian statistics, with flatness  $F_4 \approx 2.85$ , below the Gaussian value  $F_4 = 3$  (Jiménez *et al.* 1993; Noullez *et al.* 1997). The difference is small and, in the absence of theoretical reasons to suspect otherwise, it is usually attributed to experimental uncertainty. Batchelor (1953, p. 170), for example, uses an experimental flatness of  $F_4 \approx 2.86$  to support being the velocity p.d.f. Gaussian. Vincent & Meneguzzi (1991) also quote their velocity fluctuations as being ‘close’ to Gaussian, but the accompanying plot is clearly sub-Gaussian. Anselmet *et al.* (1984) note that the far tails of their velocity distribution do not behave as Gaussian and appear to be bounded beyond approximately five standard deviations, but their p.d.f. (in a duct off the central plane) is highly skewed and difficult to use in the present context.

The purpose of this note is to show that sub-Gaussian behaviour is a natural consequence of the steepness of the energy spectrum, and of the properties of the energy-containing eddies. Gaussianity is usually expected to be the result of the central limit theorem, which states that the sum of many independent random variables approaches a Gaussian probability distribution. This is only true under certain conditions. Essentially the distribution of each summand has to have a finite variance, and should only account for a vanishing fraction of the variance of the sum (Feller 1971, p. 262). It was already noted by Jiménez (1996) that the first condition is not satisfied by the velocity gradients in the presence of vortical structures, and that the correct application of the theorem can be used to explain quantitatively the p.d.f.s of the gradients in two-dimensional turbulence. The same idea was extended

to three-dimensional turbulence by Min, Mezić & Leonard (1996). We will show here that the second condition is only satisfied, for variables represented as sums of Fourier harmonics, when the power spectrum is less steep than  $k^{-1}$ . Otherwise, as in the  $k^{-5/3}$  case of the turbulent velocity fluctuations, the p.d.f. of the flow is dominated by those of the largest summands. Only in the particular case in which the latter are Gaussian, can the global p.d.f. be expected to be unconditionally so. A well-known model of turbulent flows, formalized for example by Tennekes & Lumley (1972), is that they are organized in terms of ‘eddies’, which are packets of Fourier modes which are coherent over sizeable regions in Fourier space. In this model the energy-containing scales can be represented as a discrete sum of harmonics with approximately uniform amplitudes, each of which has a p.d.f. which will be shown below to be sub-Gaussian, and the global p.d.f. inherits that character.

The organization of this note is as follows. The basic idea is developed for a one-dimensional discrete model in §2 and extended to the three-dimensional case in §3, where it is compared with experimental results. A final section briefly discusses the results in relation to the non-Gaussian statistics of other turbulent quantities.

## 2. A one-dimensional discrete example

Consider the p.d.f. of a random periodic function defined in  $x \in (0, 2\pi)$  as a Fourier series with a power spectrum  $E = k^\alpha$ ,

$$u(x) = \sum_1^N a_k \sin(\theta_k + kx), \quad a_k = k^{\alpha/2} = E(k)^{1/2}. \quad (2.1)$$

The continuity characteristics of series of this type, when  $N \rightarrow \infty$ , have been studied in Kahane (1968). Under the assumption that all the phases  $\theta_k$  are random, independent and uniformly distributed, the problem reduces to that of finding the p.d.f. of the sum of independent summands, each of which is the sine of a uniformly-distributed random variable,  $u = a_k \sin(\theta)$ , and has the distribution

$$p_k(u) = \frac{1}{\pi} (a_k^2 - u^2)^{-1/2}. \quad (2.2)$$

This can be done by computing the product of the corresponding characteristic functions  $\phi_k$ . We will assume from now on that all the distributions are even, in which case (Feller 1971, §15)

$$\phi_k(s) = 2 \int_0^\infty \cos(su) p_k(u) du, \quad (2.3)$$

and

$$\phi(s) = \prod_1^N \phi_k(s). \quad (2.4)$$

The composite p.d.f. is obtained by inverse Fourier transformation,

$$p(u) = \frac{1}{\pi} \int_0^\infty \cos(su) \phi(s) ds. \quad (2.5)$$

The characteristic function of (2.2) is

$$\phi_k(s) = J_0(a_k s) \quad (2.6)$$

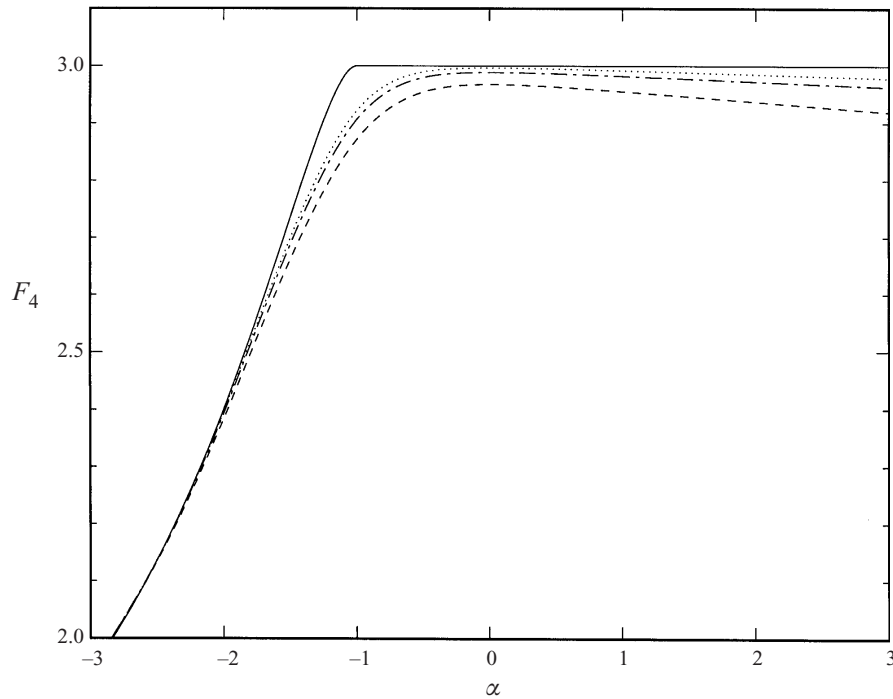


FIGURE 1. Flatness of the p.d.f. of the function (2.1), under the assumption of uncorrelated phases, as a function of spectral slope  $\alpha$ : —, analytic result for an infinite series; ·····,  $N = 120$ ; — · —,  $N = 80$ ; ----,  $N = 40$ .

where  $J_0$  is the Bessel function of the first kind. In general if the distributions of the summands differ only by a scaling factor,

$$p_k(u) = a_k^{-1} p_0(u/a_k), \quad \phi_k(s) = \phi_0(a_k s). \quad (2.7)$$

The result of the numerical evaluation of (2.4) for different values of  $N$  and  $\alpha$  is given in figure 1. The p.d.f.s are sub-Gaussian for steep spectra, and their flatness only approaches the Gaussian value  $F_4 = 3$  for an infinite number of modes and for  $\alpha \geq -1$ .

That the resulting distributions are not Gaussian even for an infinite number of terms should not be surprising since, as discussed in the Introduction, Gaussianity is only expected as a result of the central limit theorem, which applies to sums of variables which have roughly similar magnitudes. The spectrum in (2.1) destroys the equivalence of the summands, and the final distribution inherits the properties of the largest term, which is in this case sub-Gaussian. It is clear, for example, that for  $\alpha \rightarrow -\infty$  the only significant term in (2.1) would be the first one, and that the global distribution would be (2.2), whose flatness is  $F_4 = 3/2$ .

For  $\alpha = 0$  all the summands in (2.1) have the same amplitude, and the central limit theorem implies that the distribution should be Gaussian. It is less obvious that the same should be true for  $\alpha > -1$ , but shallow spectra contain many large coefficients which act like roughly equivalent variables. Steep spectra only have a few large coefficients, which dominate the rest.

The moments of  $p(u)$  can be obtained directly from the derivatives of the charac-

teristic function at  $s = 0$ . Expanding (2.3) in Taylor series gives

$$\mu_n = \int u^n p(u) du = (-)^{n/2} \phi^{(n)}(0). \quad (2.8)$$

For (2.4) at  $s = 0$ ,

$$(\log \phi)'' = \phi'' = \sum (\log \phi_k)'', \quad (2.9)$$

$$(\log \phi)^{(iv)} = \phi^{(iv)} - 3\phi''^2 = \sum (\log \phi_k)^{(iv)}, \quad (2.10)$$

where we have used that  $\phi$  is even, and that  $\phi(0) = 1$ . Using the similarity property (2.7) we obtain

$$\mu_2 = \beta_2 \sum a_k^2, \quad (2.11)$$

$$\mu_4 = 3\mu_2^2 + \beta_4 \sum a_k^4, \quad (2.12)$$

where the coefficients

$$\beta_n = (-)^{n/2} (\log \phi_0)^{(n)}(0) \quad (2.13)$$

are related to the moments of the unscaled distribution in (2.7). For the particular case (2.6),  $\beta_2 = 1/2$  and  $\beta_4 = -3/8$ . When  $\alpha < -1$  the series in (2.11) and (2.12) converge as  $N \rightarrow \infty$ , and the flatness can be expressed in terms of the Riemann zeta function,

$$F_4 = \mu_4 / \mu_2^2 = 3 \left( 1 - \frac{S_4}{2S_2^2} \right), \quad (2.14)$$

where

$$S_n = \sum a_k^n = \zeta(-\alpha n/2). \quad (2.15)$$

This is the solid curve in figure 1. For shallower slopes  $S_2$  diverges and the flatness tends to its Gaussian value  $F_4 = 3$ . For  $\alpha > -1/2$ ,  $S_4$  also diverges, but  $S_2^2 \sim N^{2+2\alpha}$  for  $N \gg 1$ , while  $S_4 \sim N^{1+2\alpha}$ , and  $F_4$  still tends to Gaussian for large  $N$ . The same is true for all the other moments. In fact the condition for the central limit theorem to hold is that the series of the individual variances,  $S_2$ , should diverge (Feller 1971, p. 262). Expressions for the hyperflatnesses up to eighth order can be found in the Appendix.

### 3. The three-dimensional case

Several corrections are needed in the argument in the previous section before it can be extended to a general multidimensional case. In the first place the Fourier coefficients  $a_k$  are now functions of a vector wavenumber, whose magnitude will be denoted by  $k$ . Next, the velocity is not periodic in space and its Fourier representation is not discrete. To express it in the form (2.1) we need to define a coherence length in wavenumber space,  $\delta(k)$ , and consider all the Fourier components within a box of size  $\delta^3$  as correlated. The sum (2.1), with the obvious interpretation of  $kx$  as an inner product, would then extend over independent wavenumber boxes. Finally, the amplitudes  $a_k$  are in general random variables themselves, of which only some characteristic value is known. This is equivalent to changing the probability density in (2.2) which, if we keep the similarity assumption for the distributions, only has the effect of changing the coefficients  $\beta_n$  in (2.11)–(2.14).

Consider the behaviour of the velocity component  $u_1$ , and define  $\psi$  to be the

polar angle in wavenumber space with respect to  $k_1$ . It follows from isotropy and incompressibility that the spectral density of  $u_1$  has the form (Batchelor 1953, p. 49)

$$\Phi_{11} = \frac{E(k)}{4\pi k^2} \sin^2 \psi, \quad (3.1)$$

so that the characteristic amplitude of the single harmonic representing each box is given by  $a_k^2 = \Phi_{11}\delta^3$ . The sums  $S_n$  can be computed by collecting coefficients over spherical shells of radius  $k$  and thickness  $\delta(k)$ . Over each shell the sum is approximated by an integral over  $\psi$  by introducing a 'surface element'  $\delta^2$ ,

$$\begin{aligned} S_n &\approx \sum_k \delta^{-2} \int_0^\pi (\Phi_{11}\delta^3)^{n/2} 2\pi k^2 \sin \psi \, d\psi \\ &\approx \frac{n!!}{(n+1)!!(4\pi)^{n/2-1}} \int_0^\infty (E(k)k^{-2}\delta^3)^{n/2} k^2 \delta^{-3} \, dk, \end{aligned} \quad (3.2)$$

where  $n!! = n(n-2)\dots 1$ . Note that the approximation of the sum over  $k$  by an integral makes this formula inaccurate if the spectrum is very steep.

It is clear that the behaviour of the moments depends as much on  $\delta(k)$  as on the form of the energy spectrum. Its meaning is that structures which contribute energy to the spectrum at wavenumber  $k$  have coherence lengths in physical space of the order of  $\delta^{-1}$ . The simplest similarity assumption, and the one that will be used here, is that  $\delta = ck$ , but other behaviours are possible. We may assume, for example, that

$$E(k) = k^\alpha, \quad \delta(k) = ck^\gamma. \quad (3.3)$$

The similarity case,  $\gamma = 1$ , is the classical eddy model of turbulence (Tennekes & Lumley 1972) in which the spectrum at each wavenumber receives contributions from structures of size  $O(k^{-1})$ , and where the proportionality constant  $c = O(1)$  is an inverse measure of the local organization; a larger  $c$  corresponds to smaller structures. The case  $\gamma < 1$  implies structureless small scales since, as  $k \gg 1$ , the coherence diameter becomes  $\delta \ll k$ , and the velocity behaves like uncorrelated noise.

In the case  $\gamma > 1$  the diameter of the spectral correlation length increases faster than the wavenumber, and eventually the whole spectrum becomes correlated. It is therefore outside our analysis, in which we model the flow as a superposition of structures which are local in spectral space. This case would occur when strong extended structures contribute significantly to the spectrum, as may be the case for the vortex filaments in isotropic turbulence (Jiménez *et al.* 1993), or for the two-dimensional discontinuities observed in the mixing of passive scalars by Pumir (1994). Localized structures correspond to Fourier components which are correlated over a large wavenumber range, and the right approach in that case is to consider the flow as a random superposition of structures in physical space (Jiménez 1996; Min *et al.* 1996). It was shown by Jiménez (1996) that, while those structures are dominant in the p.d.f.s of the gradients, they only affect weakly those of the velocities.

For  $\gamma = 1$  the substitution of (3.3) into (3.2) provides closed expressions for the  $S_n$  and for the flatnesses. Assuming for simplicity a cutoff for the spectrum at low wavenumbers,

$$E(k) = \begin{cases} k^\alpha & \text{if } k > 1 \\ 0 & \text{otherwise,} \end{cases} \quad (3.4)$$

it is easy to see that, as in the previous section, the key criterion for Gaussianity is the divergence of the integral (3.2) for the second-order moment. This happens when  $\alpha > -1$  and, in that range, all the moments take their Gaussian values as the

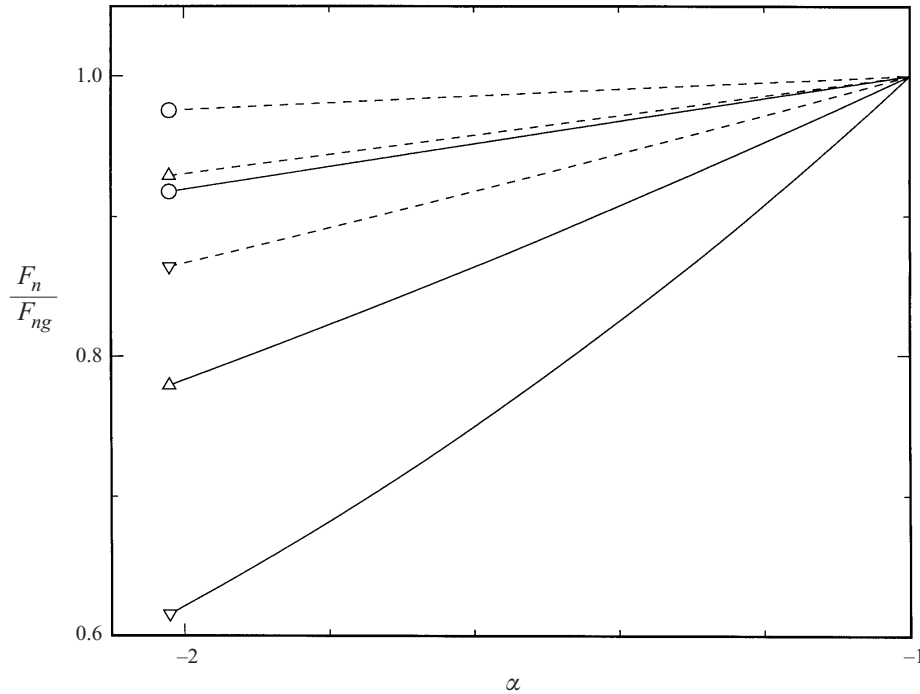


FIGURE 2. Theoretical hyperflatness from (3.5) as a function of the spectral slope, normalized with their Gaussian value:  $\circ$ ,  $n = 4$ ;  $\triangle$ ,  $n = 6$ ;  $\nabla$ ,  $n = 8$ ; ----,  $c = 1$ ; —, 1.5.

maximum wavenumber tends to infinity. The quotients that go into the formula for the hyperflatness coefficients are

$$\frac{S_n}{S_2^{n/2}} = \frac{3(n-2)!!}{(n+1)!!} \left[ -\frac{3(\alpha+1)c^3}{8\pi} \right]^{n/2-1}. \quad (3.5)$$

The result of applying this equation to the first three hyperflatnesses is shown in figure 2 as a function of  $\alpha$ , for deterministic amplitudes for which the p.d.f. of the individual harmonics is (2.2). The resulting p.d.f.s are always sub-Gaussian for  $\alpha < -1$ .

Although the approximations made in the present analysis are too rough to justify claims of quantitative accuracy, the conclusion that essentially random signals can have sub-Gaussian statistics for a spectrum steeper than  $k^{-1}$  suggest that the effect should be observable in turbulent velocity fluctuations, which have a spectrum of approximately  $k^{-5/3}$ . As mentioned in the Introduction, the velocity p.d.f.s are usually assumed to be Gaussian, lacking the spectacular tails of those of the gradients, and there are few published examples detailed enough to compute the higher moments. Two of those are shown in figure 3, each of them representing a range of Reynolds numbers for two different flows. They are both sub-Gaussian and agree approximately. In the figure we have also plotted the result of (2.2)–(3.5) for the self-similar case  $\gamma = 1$  and a coherence constant  $c = 1.5$ , which was chosen to fit the data. The agreement is fair.

The details of the low-wavenumber end of the spectrum are relatively unimportant for the previous argument as long as the integrals converge. Consider, for example, a

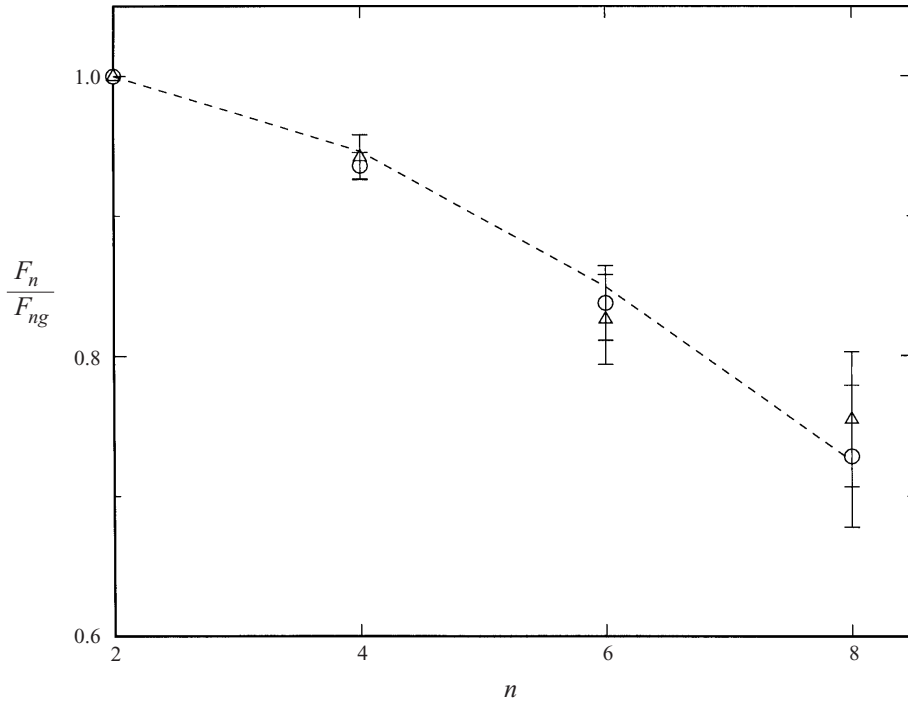


FIGURE 3. Hyperflatness of measured p.d.f.s of velocity fluctuations. Error bars are ranges of measurements for different Reynolds numbers.  $\circ$ , Numerical periodic box,  $Re_\lambda = 63-165$  (Jiménez *et al.* 1993).  $\triangle$ , Round jet,  $Re_\lambda = 360-600$  (Noullez *et al.* 1997). Dashed line is (3.5) with  $\delta/k = 1.5$ .

slightly more general form than the one in (3.4),  $E(k) = k_0^{-1} F(k/k_0)$ , where

$$F(k) = \begin{cases} k^\alpha & \text{if } k > 1 \\ k^\beta & \text{otherwise,} \end{cases} \quad (3.6)$$

with  $\beta > 0$ . The integral in (3.2) converges for  $\alpha \leq -1$ , and it is easy to see that the result is equivalent to substituting  $(\beta - \alpha)(\alpha + 1)/(\beta + 1)$  for  $(\alpha + 1)$  in (3.5). For a given spectrum, the effect is equivalent to choosing a different value for  $c$ , and figure 3 is unaffected.

#### 4. Discussion and conclusions

The main conclusion of this note is that Gaussian distributions should not in general be expected in homogeneous turbulent flows for variables whose spectra are steeper than  $k^{-1}$ . The velocity fluctuations, whose spectrum is approximately  $k^{-5/3}$ , or steeper in the dissipative range, fall in that category. The behaviour in those cases depends on the organization in the flow. We have analysed here the case in which the organization is spectral, so that the velocity can be modelled as a discrete sum of independent Fourier components. In those cases, for steep spectra, the low wavenumbers dominate the sum and the fluctuations inherit their distribution. The equation (2.14) for the flatness can be rewritten in general as

$$F_4 = 3 + (F_{40} - 3)S_4/S_2^2, \quad (4.1)$$

where  $F_{40}$  is the flatness of each individual summand, so that sub-Gaussian summands generate sub-Gaussian distributions and vice versa. In essence, the turbulent velocity spectrum is not ‘random enough’ for the central limit theorem to apply. For shallower spectra the model predicts Gaussian distributions, at least as long as the individual summands have bounded variances.

In the opposite type of organization the structures are global in spectral space and spatially local. The present analysis is not then valid but a partial treatment can be found in Jiménez (1996). The approximation is then that the velocity is the superposition of random localized physical structures. The final distribution depends on the relative frequency of structures having a given amplitude. Because the behaviour of the tails of the p.d.f.s associated to the individual structures is connected with the total volume filled with velocities or gradients above a given magnitude, they tend to decay only algebraically, and the resulting distributions are super-Gaussian. This behaviour is not connected with the power spectrum.

The limit between the two cases is the classical ‘eddy’ model (Tennekes & Lumley 1972), in which structures of size  $O(k^{-1})$  correspond to wave packets with wavenumbers in the range  $k \pm O(k)$ . This limit falls within the present analysis, and we have treated in detail the case in which the coefficients of each harmonic are deterministic, while the phases are random. The p.d.f. of each component is then that of a random sine (2.2), with  $F_{40} = 3/2$ , and the resulting p.d.f.s are therefore sub-Gaussian. The results are used in figure 3, and the best fit with the available experimental data for the velocity fluctuations is achieved by assuming eddies with spectral diameter  $\delta \approx 1.5k$ . The flatness is then  $F_4 \approx 2.85$ , and agrees well with the flows for which we could find reliable data. Given our previous discussion on the several factors that might be important for the final result, and especially since the final p.d.f. would be dominated by the non-universal large scales of the flow, there is no guarantee that this distribution is universal, and other flows should be carefully studied. The present analysis suggests, however, that the p.d.f. of the velocity fluctuations should be a source of information on the structure of the large scales of turbulent flows comparable to that of the gradients for the small ones.

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## Appendix. Higher-order moments

The expressions for the higher-order derivatives of the logarithm are

$$(\log \phi)^{(6)} = \phi^{(6)} - 15\phi^{(4)}\phi^{(2)} + 30\phi^{(2)^3}, \quad (\text{A } 1)$$

$$(\log \phi)^{(8)} = \phi^{(8)} - 28\phi^{(6)}\phi^{(2)} - 35\phi^{(4)^2} + 420\phi^{(4)}\phi^{(2)^2} - 630\phi^{(2)^4}, \quad (\text{A } 2)$$

from which the corresponding hyperflatness,  $F_n = \mu_n/\mu_2^{n/2}$ , can be written as

$$F_6 = -30 + 15F_4 + \beta_6 S_6/\beta_2^3 S_2^3, \quad (\text{A } 3)$$

$$F_8 = 630 + 28F_6 + 35F_4^2 - 420F_4 + \beta_8 S_8/\beta_2^4 S_2^4. \quad (\text{A } 4)$$

For the particular case of (2.6),

$$\beta_2 = 1/2, \quad \beta_4 = -3/8, \quad \beta_6 = 5/4, \quad \beta_8 = -1155/128. \quad (\text{A } 5)$$



When  $S_2$  diverges the corrections vanish, and the hyperflatnesses take their Gaussian values  $F_{2n} = (2n - 1)!!$ .

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