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# Intermittency and cascades

By JAVIER JIMÉNEZ

School of Aeronautics, Universidad Politécnica, 28040 Madrid, Spain and Centre for Turbulence Research, Stanford University, Stanford, CA 94305, USA

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The theory of intermittency in multiplicative cascades is reviewed, with special, but not exclusive, emphasis on its applications to turbulence. It is noted that, in many physical systems, this theory is incomplete, and two of its limitations are discussed in some detail. It is first argued that large fluctuations will in most cases behave differently from the lower-level background, since the overall mean introduces an intensity scale that breaks self-similarity, and that they will, under the right conditions, evolve into coherent structures decoupled from the rest of the system. The effect of non-local interactions is then addressed. It is shown that the results depend on the nature of the interaction, and that it is possible to generate non-local cascades which are less intermittent, as intermittent, or even more intermittent, than local ones. It is finally stressed that the multiplicative theory of cascades is a kinematic description, and that its relation with the real dynamics is not straightforward.

## 1. Introduction

Perhaps one of the most useful tools in the analysis of complex systems is the idea of the cascade, which was probably first explicitly introduced by Richardson (1922) to describe high-Reynolds-number turbulence. Kolmogorov (1941) made it quantitative, on the implicit assumption that the magnitude of the fluctuations was small enough for all the points in the flow to be described in terms of uniform characteristic scales. His later introduction of intermittency corrections (Kolmogorov 1962) improved on this approximation, but still assumed a uniform cascade in the sense that the only effect of local fluctuations was to introduce a locally variable velocity scale. We will see below that more complicated effects are possible.

Intermittency is a common phenomenon in many complex systems, and we will see that it is a natural consequence of cascades, in the same way that exponentially growing solutions are properties of linear ordinary differential equations. As in that case, the simplest theory leads to infinities and, also as in differential equations, it is limited in the real world by the appearance of nonlinearity and of spatial effects. We will explore in this paper some of the limits of the theory, and how they manifest themselves in physical systems. Because of the personal bias of the author, most of the examples will be drawn from Navier–Stokes turbulence, but the mathematics are more general, and there is nothing in them that necessarily implies eddies, or even an underlying geometry. The theory of probability and stochastic processes is of course one of the most abstract branches of mathematics, and the literature on it is enormous. No attempt will be made to survey it here, but most of what we discuss could be seen as a particular example of it.

The purpose of the present paper is to summarize the theory of intermittency, and to point to some of the directions in which it needs extension and modification. The classical theory of self-similar multiplicative cascades, and the resulting appearance of multifractals, are discussed first. We next observe that, in most physical systems, this theory is bound to fail for very strong fluctuations, since the mean value of the cascading variable introduces an extra scale, and large fluctuations eventually decouple from the low-level background and follow different laws. We next look at non-local effects, such as those in inverse cascades, in which the evolution of an element depends on more than one precursor. We give arguments and examples to show that these cascades cannot be assumed *a priori* to be intermittent, but that many behaviours are possible. Finally, some remarks are made on the kinematic character of these descriptions, and on their relation with the underlying dynamics.

## 2. Multiplicative cascades

Multiplicative cascades have been proposed in several contexts, probably, like much of probability theory, first in economics and social sciences (see Bartholomew 1973 and the references in Amaral *et al.* 1998). They have been the subject of a lot of work. A recent survey of their applications in physics is Paladin & Vulpiani (1987), and semi-popular but rigorous accounts of their properties and of their appearance in other fields have been given by Mandelbrot (1983) and Schroeder (1991). They were first applied to turbulent flows by Gurvich & Yaglom (1967), and in more detail by Novikov (1971, 1990), although they were already implicit in the original paper on intermittency by Kolmogorov (1962). Recent reviews have been given by Meneveau & Sreenivasan (1991), Nelkin (1994) and Sreenivasan & Stolovitzky (1995). We will summarize here the properties of multiplicative cascades in their simplest form, assuming for example that all the statistical distributions can be described by density functions.

Assume a locally deterministic cascade, defined in this paper as one in which the probability distribution of the cascading variable, $p_n(u_n)$, depends only on its value at the previous step,

$$p_{n+1}(u_{n+1}) = \int p_t(u_{n+1}|u_n;\, n)p_n(u_n)\,\mathrm{d}u_n. \tag{2.1}$$

This is in contrast to the p.d.f. having a more complicated functional dependence, such as on the values of $u_n$ in some extended spatial neighbourhood, or on several previous cascade stages. This assumption intuitively implies that $u_{n+1}$ evolves faster, or on a smaller scale, than $u_n$, and is in some kind of equilibrium within its precursor. If the cascade is deterministic in this sense, $u_n$ can be represented as a product

$$u_n = x_n x_{n-1} \dots x_1 u_0 \tag{2.2}$$

of factors, $x_n = u_n/u_{n-1}$, which are statistically independent of each other.

If moreover the underlying process is invariant to scaling transformations, it should also be true that the transition p.d.f. has the form

$$p_t(u_{n+1}|u_n) = u_n^{-1} w(u_{n+1}/u_n;\, n). \tag{2.3}$$

The multiplicative model works most naturally for positive variables, and we will assume that to be the case in this paper, but most results can be generalized to arbitrary distributions with suitable care.

Local deterministic self-similar cascades lead naturally to intermittent distributions, in the sense that the high-order flatness factors for $u_n$ become arbitrarily large as $n$

increases. It follows from (2.1)–(2.2) that the $\lambda$th order moment for $p_n$ can be written as

$$S_n(\lambda) = \int \xi^{\lambda} p_n(\xi)\,\mathrm{d}\xi = S_0(\lambda) \prod_{j=1}^{n} S_w(\lambda;\, j), \tag{2.4}$$

where $S_w(\lambda;\, j)$ is the $\lambda$th order moment of the $j$th factor $x_j$, and $\lambda$ is any real number for which the integral exist. If we define flatness factors as

$$F(\lambda) = S(\lambda)/S(1)^{\lambda}, \tag{2.5}$$

(2.4) is equivalent to

$$F_n(\lambda) = F_0(\lambda) \prod_{j=1}^{n} F_w(\lambda;\, j). \tag{2.6}$$

Chebichev's inequality implies that

$$S(\lambda) \geqslant S(\lambda-1)S(1) \geqslant S(\lambda-2)S^2(1)\ldots, \tag{2.7}$$

with equality only holding for the degenerate case of distributions concentrated on a single point, from where

$$1 \leqslant F(2) \leqslant F(3)\ldots. \tag{2.8}$$

The product in (2.6) therefore increases without bound with the number of cascade steps. Note that the flatness defined in (2.5) is uncentred, while experiments normally use centred moments. It is easy to see that the conclusion would be the same for them, since the dominant term in the expansion of the centred flatness is always the uncentred flatness of highest order. High flatnesses imply long tails in the p.d.f., representing strong uncommon events, and are the hallmarks of intermittency.

We have up to now left open the possibility that the p.d.f.s $w(x_n)$ of the individual factors depend explicitly on the cascade step $n$. Most of the properties of multiplicative cascades carry over to that case, but they are expressed most simply if all the steps are equivalent, in which case (2.4) and (2.6) are power laws

$$S_n(\lambda)/S_0(\lambda) = S_w(\lambda)^n, \qquad F_n(\lambda)/F_0(\lambda) = F_w(\lambda)^n. \tag{2.9}$$

To simplify the notation, we will from now on make that assumption. It is mathematically tempting to substitute $n$ in (2.9) by a continuous variable, in which case the p.d.f.s form a continuous semigroup generated by an infinitesimal scaling step. This leads to a beautiful theory, summarized in Novikov (1990), but it is not necessarily a good idea from a physical point of view. For example, while there might be some justification for assuming that the distribution of the properties of an eddy of size $h$ depend only on the eddy of size $2h$ from which it derives, the same argument is weaker between eddies of almost equal sizes. We will restrict ourselves here to the discrete case.

### 2.1. *Limiting distributions*

The process described above can be summarized as a family of distributions $p_n(u_n)$ such that the probability density for the product of two variables is

$$p(u_{n_1}u_{n_2}) = p_{n_1+n_2}(u_{n_1+n_2}), \tag{2.10}$$

and it is natural to ask whether such a process has a limit for large $n$. The general question is addressed in Feller (1971) for sums of variables, rather than for products.

The first attempts were therefore to reduce the product to a sum by defining

$$z = n^{-1} \log (u_n/u_0). \tag{2.11}$$

The argument was that, since sums of independent variables tend to Gaussian distributions under fairly general conditions (Feller 1971, p. 578), the same would be true for $z$, and the limiting distribution for $u_n$ would be lognormal (see Gurvich & Yaglom 1967, or the original proposal for lognormality in Kolmogorov 1962). This reasoning was shown to be incorrect by Novikov (1971) who noted that, even if the central part of the distribution might approach a lognormal, the tails do not. The family of lognormal distributions is a fixed point of (2.10), but it is unstable and does not appear as a limit except for initially lognormal generators. It is characterized by moments of the form

$$S_w(\lambda) = \exp (a\lambda + b\lambda^2), \tag{2.12}$$

which is conserved under (2.9), so that the product of lognormally distributed variables stays lognormal. The moments (2.12) are generated by the recursive relation

$$Q_w(\lambda) = \frac{S_w(\lambda+3)S_w^3(\lambda+1)}{S_w(\lambda)S_w^3(\lambda+2)} = 1, \tag{2.13}$$

with suitable initial conditions for $\lambda < 2$. Under (2.9), $Q_n(\lambda) = Q_w^n(\lambda)$, and it is clear that only when all the $Q(\lambda)$ are initially equal to 1 do they continue to be so under multiplication. Otherwise, any $Q_w$ initially larger than 1 would tend to infinity after enough cascade steps, while any one initially smaller than 1 would tend to zero. Only an exactly lognormal distribution of the generating factors gives rise to a lognormal distribution, and even small errors lead to very different patterns of moments. It is clear from this argument that this instability is shared by other families of distributions, in contrast to the situation for sums of random variables, in which the Gaussian distribution is not only a fixed point, but one with a very large basin of attraction. Since the lognormal distribution is frequently quoted as the result of multiplicative cascades, the details of how the instability manifests itself are given in the Appendix.

### 2.2. *Multifractals*

It should be clear from the previous discussion that the problem is not the technique of analysing the statistics of products in terms of those of sums, but the inappropriate use of the central limit theorem. The general form of the limiting distributions can be derived from the theory of large deviations of sums of random variables. A detailed account is found in Lanford (1973) and a useful summary and discussion in the book by Frisch (1995, pp. 168–171). The result is that, for $n \gg 1$,

$$p_n(u_n) \approx \left(\frac{-\phi_0''}{2\pi n}\right)^{1/2} \mathrm{e}^{n[\phi(z)-z]}, \tag{2.14}$$

where $z$ is defined as in (2.11) and $\phi$, which plays the role of an entropy, is a smooth function of $z$, concave downwards. Primes stands for derivatives with respect to $z$. We define $z_\lambda$ as the point where

$$\phi_\lambda' \equiv \phi'(z_\lambda) = -\lambda, \tag{2.15}$$

which corresponds to the location of the maximum of $\phi + \lambda z$. The condition that $p_n$ should have unit mass reduces to $\phi_0 = 0$, after using Laplaces's method to expand $\int p_n \, \mathrm{d}u_n$ for $n \gg 1$ (Bender & Orszag 1978).

The entropy $\phi$ can be related directly to the moments of the transition probability density. Using again Laplace's method to expand the $\lambda$th moment of $p_n$,

$$S_n(\lambda) = \int_{-\infty}^{\infty} n\mathrm{e}^{n(\lambda+1)z} p_n(u_n)\,\mathrm{d}z \approx \left(\frac{\phi_0''}{\phi_\lambda''}\right)^{1/2} \mathrm{e}^{n(\phi_\lambda+\lambda z)}, \tag{2.16}$$

from where, using (2.9),

$$\sigma_\lambda \equiv \log S_w(\lambda) = \phi(z_\lambda) + \lambda z_\lambda. \tag{2.17}$$

Note that the essence of Laplace's method is that, for $n \gg 1$, most of the contribution to the integral (2.16) comes from the neighbourhood of $z_\lambda$, so that it makes sense to consider each such neighbourhood as a separate 'component' of the cascade.

The classic geometric interpretation of this representation as a multifractal was given by Parisi & Frisch (1985) for the case of three-dimensional homogeneous turbulence. We have assumed little, up to now, about the nature of each cascade step, but it is natural in turbulence to interpret it as the process in which eddies decay to a smaller geometric scale, that we will assume to be $\mathrm{e}^{-1}$ times the original one. The argument works for any variable for which scale similarity can be invoked, but most experiments are done for the magnitude of the velocity increments across a distance $\ell$ which, in this interpretation, would be $\ell_n/\ell_0 = \mathrm{e}^{-n}$. The pair (2.11) and (2.14) can then be written

$$u_n/u_0 = (\ell_n/\ell_0)^{-z_\lambda}, \qquad p_n(z_\lambda) \sim (\ell_n/\ell_0)^{-\phi_\lambda}. \tag{2.18}$$

The multifractal interpretation is that the 'component' indexed by $\lambda$, whose velocity increments are 'singular' in terms of $\ell$ with exponent $z_\lambda$, lies on a fractal whose volume is proportional to its probability, and which therefore has a fractal dimension $D(z_\lambda) = 3 + \phi_\lambda$. Note that the previous discussion implies that the multifractal component $\lambda$ dominates the structure function $S_n(\lambda)$, which can be expressed as

$$S_n(\lambda) \sim (\ell_n/\ell_0)^{\sigma_\lambda}, \tag{2.19}$$

and that the scaling exponents $\sigma_\lambda$, the multifractal spectrum $D(z_\lambda)$, the transition probability distribution $w(x)$, and the limiting distribution $p_n(u_n)$, univocally determine each other.

Some properties can be easily derived from the previous discussion. It follows from (2.15)–(2.17) that

$$z_\lambda = \mathrm{d}\sigma_\lambda/\mathrm{d}\lambda, \tag{2.20}$$

and from (2.17) that $\sigma_\lambda$ is concave downwards. If we assume, for example, that the multiplicative factor $x$ is bounded above by $x_b$, which is reasonable for many physical systems, (2.11) implies that $z_\lambda \leqslant \log x_b$. In fact, if the transition probability behaves near $x_b$ as $w(x) \sim (x_b - x)^\beta$ the scaling exponents tend, for $\lambda \gg 1$, to

$$\sigma_\lambda = \lambda \log x_b - (\beta+1)\log\lambda + O(1). \tag{2.21}$$

The case in which the logarithmic term is missing, which implies a distribution with a concentrated component $\delta(x - x_b)$, has been explored by Meneveau & Sreenivasan (1991) and by She & Leveque (1994). In all cases the singularity exponent of the set associated with $\lambda \to \infty$ is $z_\infty = \log x_b$. In the case of a concentrated distribution the dimension of this set approaches a finite limit, but otherwise

$$D(\lambda) \approx -(\beta+1)\log\lambda, \tag{2.22}$$

which becomes infinitely negative. This should not be considered a flaw. The set of events which only happen at isolated points and at isolated instants has dimension $D = -1$ in three-dimensional space, and those which only happen at isolated instants, and only under certain circumstances, have still lower negative dimensions.

The multifractal spectrum of velocity differences in three-dimensional Navier–Stokes turbulence has been measured for several flows in terms of the scaling exponents of the structure functions, and appears to be universal. A review is given by Meneveau & Sreenivasan (1991), and more recently by Frisch (1995, § 8). Van Atta & Yeh (1975) and Chhabra & Sreenivasan (1992) measured directly the transition probability of the multipliers, which agree well with the spectra obtained from the exponents, independently of the geometric scale factor assumed for each step. This evidence will be further discussed in the next section. There has been extensive theoretical work on the consequences of imposing various physical constraints on the multipliers. Novikov (1994) explored the properties of conservative cascades in which the average value of the variable is conserved, with the energy dissipation rate in mind, while Frisch (1995, pp. 133–135) has formulated conditions for $u_n$ to remain finite as $n$ increases.

Examples of multifractal behaviour in systems different from Navier–Stokes turbulence, for which the mathematical treatment in this section applies, but where the geometric interpretation is hard to justify, are given in the already mentioned book by Schroeder (1991). An example in which the yearly sales of commercial companies are modelled as a multiplicative process, and fitted to real-world statistics, is given by Amaral *et al.* (1998). The cascade steps, in that case fiscal years, carry no implication of geometric scale.

## 3. Blocking

We have not considered up to now the reason why cascades are random, but it is likely to be due, at least in part, to the interactions among different parts of the system. The probability distribution of the multiplicative factors would in that case be determined by the result itself.

Consider for example a cascade in an extended system formed by individuals (or eddies), where $p_n(u_n)$ can be interpreted as the fraction of the total population having a given value. An individual lives in a random environment, and cascades under the influence both of its internal dynamics and of the magnitude of the ambient fluctuations.

Assume, since our variables are positive, that we take as a characteristic magnitude of the fluctuations their mean $S_n(1) = S_w(1)^n$. In general this will be of the same order of magnitude as the intensity of most eddies, and the internal and external instability mechanisms will be proportional to each other. This was the original basis for assuming self-similarity.

This argument does not apply uniformly to all individuals. Intermittency implies that some of them would be strong enough to be unaffected by fluctuations. If we define the magnitude of strong fluctuations as $S_n(\lambda)^{1/\lambda}$ for some large value of $\lambda$, it follows from (2.6)–(2.9) that $S_n(\lambda)^{1/\lambda}/S_n(1)$ increases without limit as the cascade proceeds.

There will therefore be two classes of individuals, those that cascade due to ambient fluctuations, and those that do because of their internal instabilities. It is not necessarily true that both classes follow the same physics, and therefore that the assumption of a uniformly self-similar cascade is justified. If, for example, the dynamics of the system is such that individuals are linearly stable, although not nonlinearly so, the

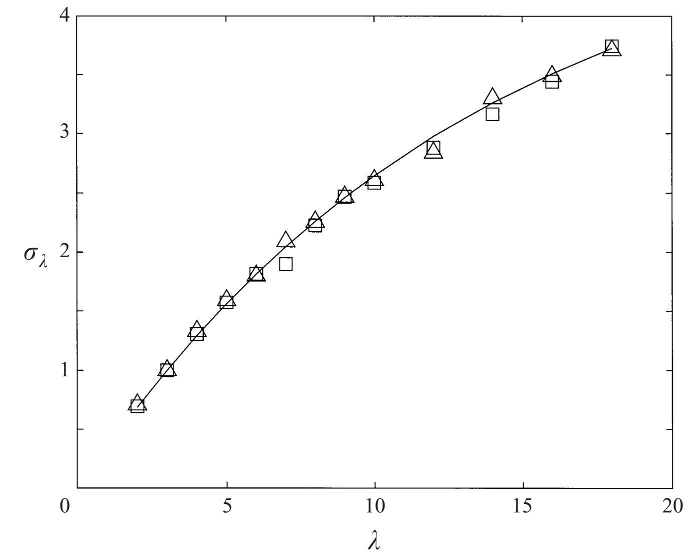


FIGURE 1. Scaling exponents of the longitudinal velocity structure functions. Symbols are experiments: □, Herweijer & van de Water (1995); △, Anselmet *et al.* (1984). The microscale Reynolds number is $Re_\lambda \approx 500$–$800$ in both cases. ———, Binomial multiplicative model (3.4), with $\alpha = 1.21$.

strongest elements will uncouple from the background perturbations, and part of the cascade will be blocked.

We can, for example, think of individuals as competitors in a marketplace. Competition among comparable players leads to a dynamic equilibrium in which each one is controlled by the interaction with the others. Occasionally, however, an individual may get strong enough to shelter itself from competition, and becomes the functional equivalent of a monopoly. Its evolution would then be governed by laws which are different from those of the rest of the system. The details depend on the particular physics involved.

### 3.1. *Three-dimensional turbulence*

Consider three-dimensional Navier–Stokes turbulence. Meneveau & Sreenivasan (1991) observed that the multifractal spectra obtained in experiments could be approximately fitted by a simple binomial multiplicative process for the energy dissipation rate $\epsilon$. The eddies in the inertial range are bisected at each step, and each half receives an unequal part of the dissipation according to fixed coefficients. The process is such that the total dissipation is conserved, so that one half of the eddy receives a fraction $\alpha > 1$, and the other one $2-\alpha$. A full cascade step consists of three bisections along the three coordinate planes so that, after a full step, the geometric scale $\ell$ has been divided by 2.

Dissipation is not a good variable to determine the stability of the different eddies, whose eigenvalues are, on dimensional grounds, proportional to the velocity gradient $\omega = (\epsilon/\ell^2)^{1/3}$. In each eddy bisection the gradients of the two halves are multiplied by $\alpha^{1/3}2^{2/9}$ and $(2-\alpha)^{1/3}2^{2/9}$. This is still a multiplicative cascade, but no longer a conservative one. The mean gradient increases after each step.

Each eddy can be characterized by its multiplicative history, and indexed by the number $k$ of $\alpha^{1/3}$ factors that it contains. Its gradient after $n$ full steps is then

$$\omega_n(k) = 2^{2n/3}\alpha^{k/3}(2-a)^{(3n-k)/3}, \tag{3.1}$$

and the probability of $k$ is

$$p_n(k) = 2^{-3n} \binom{3n}{k}, \quad k = 0 \ldots 3n, \tag{3.2}$$

where the parentheses represent the binomial coefficient. It follows from (2.9) that the mean gradient after $n$ full steps is

$$\langle \omega_n \rangle = 2^{-7n/3} [\alpha^{1/3} + (2-a)^{1/3}]^{3n}, \tag{3.3}$$

and that the scaling exponents of the structures functions, $\langle \Delta u_n^{\lambda} \rangle = \langle (\epsilon_n \ell_n)^{\lambda/3} \rangle$, are

$$\sigma_{\lambda} = \frac{\lambda}{3} - 3 \log_2 \left[ \frac{\alpha^{\lambda/3} + (2-a)^{\lambda/3}}{2} \right], \tag{3.4}$$

which agree well with experimental results if $\alpha \approx 1.21$ (figure 1). Note that the binomial distribution used in this example is of the bounded type discussed in (2.21)–(2.22). We saw then that the asymptotic behaviour of the scaling exponents depends on whether the upper limit of the p.d.f. of the generating factor is a delta function or an analytic zero. In the first case the exponents approach a linear asymptote for large $\lambda$, while in the second they behave logarithmically. The distribution used here for each partial bisection step is formed by two delta functions. For a full step, the p.d.f. is obtained by substituting $n = 1$ in (3.2). It is also bounded and formed by six delta functions. In both cases we can then expect a linear behaviour of the exponents for large $\lambda$, and a finite limiting fractal dimension. The p.d.f.s found by Chhabra & Sreenivasan (1992) are however not discrete. They look tent-like for a scale factor of two, and would therefore lead to a logarithmic behaviour of the exponents. The goodness of the fit in figure 1, even if the assumed p.d.f. is very different from the experimental one, highlights the difficulty of deducing the distribution of the factors from a finite number of measured moments. Sreenivasan & Stolovitzky (1995) noted that the agreement in this case is largely accidental, since both p.d.f.s, even if very different, have similar moments up to at least $\lambda \approx 8$. It is probably experimentally impractical to measure structure functions of much higher orders than those given in the figure (Jiménez 1998), and it is clear from the previous discussion that these are not high enough to decide which is their true asymptotic behaviour. Any discussion of the fundamental physics of the cascade should therefore be based as much as possible on actual measurements of the generating factors, rather than on the scaling exponents alone. This observation was first made by Nelkin (1995).

When this is done, problems appear. The early measurements of Van Atta & Yeh (1975) and Chhabra & Sreenivasan (1992) established that the overall probability distribution of the generating factors is independent of the scale, but later ones by Sreenivasan & Stolovitzky (1995) and Predizzetti, Novikov & Praskovsky (1996) showed that they are not independent among levels. The distribution of the multipliers depends on the geometric location of the descendent eddy within its parent, and on whether the latter was generated by a high or by a low multiplier. The latter dependence has even been shown to extend across several cascade levels. Sreenivasan & Stolovitzky (1995) considered the influence of this lack of statistical independence on the final multifractal properties of the cascade, and showed that, at least for a particular model, it is slight and only noticeable on structure functions with $\lambda < 0$. Nelkin & Stolovitzky (1996) also mentioned the weak influence of the lack of statistical independence on the behaviour of the moments. Any claim of self-similarity based on the experimental scaling behaviour of a few moments should be viewed in the light of these examples.

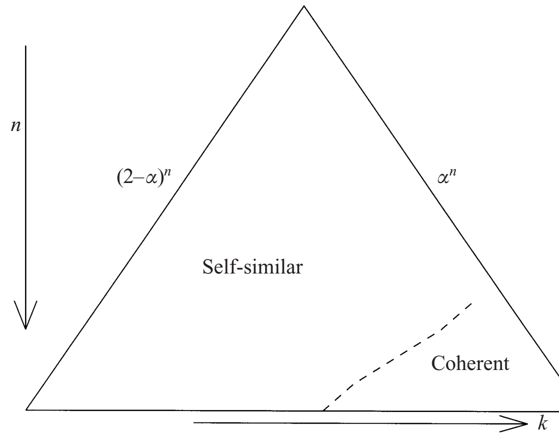


FIGURE 2. Sketch of the bisection cascade discussed in the text. The dashed line is the boundary between the multifractal and coherent ranges. $C = 1$.

Returning to the question of the local blocking of the cascade, there are several possible definitions of a 'strong' eddy. In three-dimensional turbulence the highest gradients are those at the smallest scales, which lie at the final step of the cascade, $n = N$. We will therefore define a strong eddy as one whose characteristic gradients are higher than a low multiple of the mean gradient at the final cascade stage,

$$\omega_n(k) > C\,\langle\omega_N\rangle. \tag{3.5}$$

In figure 2 we have sketched the binomial cascade discussed in this section. The index of the eddies runs from left to right, and the cascade runs from top to bottom. The range of indices grows in successive steps. At the right-hand edge lie the strongest possible eddies, those whose factors are all equal to $\alpha$, while the weakest ones form the left-hand edge. The dashed line is the coherence boundary, defined here as $C = 1$. To its right lie eddies with gradients higher than the overall mean, and which therefore have a chance of becoming coherent. The figure has been drawn for $N = 15$, whose range of scales corresponds roughly to a microscale Reynolds number $Re_\lambda = 10^3$, but it is essentially independent of $N$. The coherent range forms in the bottom third of the cascade steps and, in the last stage, fills almost half of the possible indices.

The very strong gradients in that region are created intermittently and decouple from the overall multifractal cascade. Strong isolated vorticity tends to roll up into filaments, which are stable to weak perturbations, and we might therefore expect the formation of coherent columnar vortices lasting long times. Very strong long-lived vortices have been observed in turbulent flows (see e.g. Jiménez *et al.* 1993, and references therein), and are partly responsible for the recent resurgence of interest in intermittency.

These elongated coherent objects are difficult to classify into cascade steps. Their diameters are of the order of the dissipative scale, while their lengths are comparable to the integral scale of the flow, and they thus span the full range of the cascade (Jiménez & Wray 1998). Their long lifetimes, which are of the order of the integral time scale, also suggest that they are themselves not cascading. Several attempts were made by Jiménez (1998) to fit them into the overall spectrum of multifractal dimensions, but the result is that their probability is much higher than it should be, for any reasonable interpretation of their singularity exponent. This is clearly

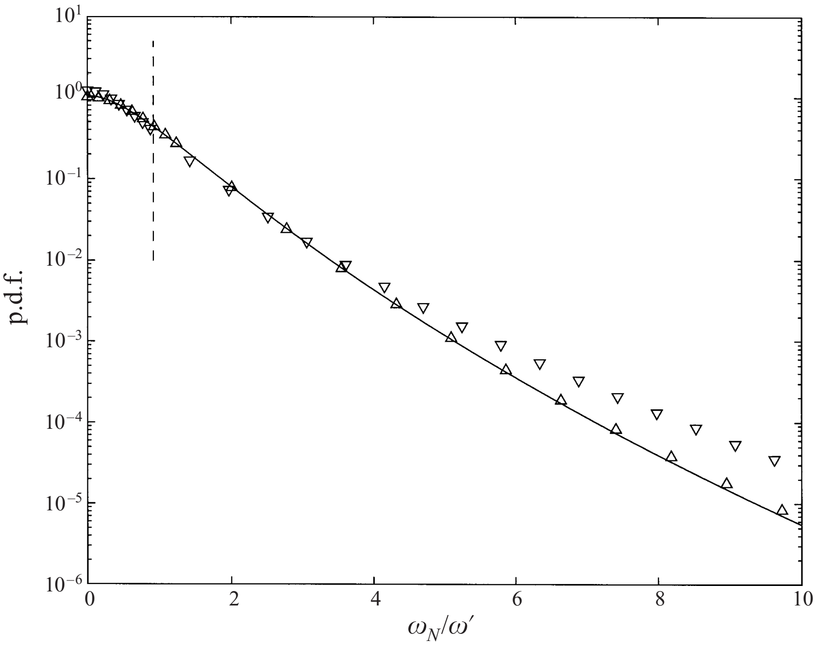


FIGURE 3. P.d.f. of the velocity gradients at the bottom of the multifractal cascade (3.2), normalized with its standard deviation. The vertical dashed line marks the left-hand boundary of the coherent range. $N = 12$, $C = 1$. Symbols are experimental gradients from numerical simulations by Jiménez & Wray (1998): $\triangle$, absolute value of the longitudinal velocity gradients, $|\partial u/\partial x|$; $\nabla$, transverse velocity gradients. $Re_\lambda \approx 170$.

connected with their lifetime. The salient feature of the observed vortex filaments is their intensity. They contain velocity differences $O(u')$ across distances which are only $O(\eta)$, and their vorticity is much stronger than the average magnitude of the surrounding velocity gradients. They are therefore essentially unperturbed, and lie beyond the self-similarity of the cascade.

That they constitute an unavoidable blocked coherent component of the cascade, as opposed to the unstructured multifractal background, was first proposed by Jiménez & Wray (1998).

An example of the p.d.f. of the gradients in the binomial model is given in figure 3. In computing the p.d.f. it is important to take into account the probability distribution at the root of the cascade, as well as the multiplicative process itself. The result in (3.2) corresponds to a concentrated initial distribution. In the general case in which the initial distribution is $p_0(\omega)$,

$$p_n(\omega) = \sum_{k=0}^{3n} \omega_n(k)^{-1} p_0[\omega/\omega_n(k)]\, p_n(k), \tag{3.6}$$

where $p_n(k)$ and $\omega_n(k)$ are taken from (3.1)–(3.2). Figure 3 has been computed using a Gaussian distribution for $p_0$, which is approximately true for the velocity increments in the largest flow scales. The p.d.f. is not symmetric and recalls those in experimental and numerical turbulent flows (see Jiménez 1998). The figure includes the limit of the coherent region, and experimental gradients from a particular numerical simulation. The agreement is partly artificial. The number of cascade steps has been optimally fitted to $N = 12$, and implies a scale ratio of $2^{12} \approx 4 \times 10^3$, while the real range of length scales at the Reynolds number of the simulation is at most $L_\varepsilon/\eta \approx 300$. The shapes of the longitudinal and transverse p.d.f.s are different, and fitting the latter to the present model would require an even larger number of steps ($N \approx 18$). This disparity between longitudinal and transverse velocity increments has been known for some time, and Dhruva, Tsuji & Sreenivasan (1997) observed that it is one of the reasons to suspect that self-similar multiplicative processes are not the whole story behind intermittency in three-dimensional turbulence.

It is seen in the figure that, even if the coherent region occupies a relatively small part of the sketch in figure 2, and accounts for a small fraction of the mass in the distribution, it is responsible for most of the intermittent tail of the gradients. That is also the case for the p.d.f.s of the gradient and of the vorticity magnitude in numerical flows. Jiménez *et al.* (1993) and Jiménez & Wray (1998) showed that removing the coherent vortex structures removed most of the intermittency, and a similar result was obtained by Belin *et al.* (1996) in experiments at higher Reynolds numbers. The location of the coherence boundary is to some extent arbitrary. If we had chosen a higher value of the proportionality constant $C$ in (3.5), the boundary would have moved proportionally to the right.

It should be emphasized that the multifractal analysis is only valid to the left of the coherence boundary, and can therefore only be used to estimate the location of the boundary itself. Inside the coherent region the cascade might still be self-similar, although the discussion in Jiménez & Wray (1998) suggests that it is probably not, but follows different rules from those of the fluctuations in the background.

### 3.2. *Two-dimensional turbulence*

The effect of blocking on the overall flow dynamics can be more dramatic than in the previous case. Consider two-dimensional turbulence. It is generally agreed that there are two possible cascades (Kraichnan 1967): an enstrophy cascade to smaller scales, and an energy cascade to larger ones. Nakamura, Takahashi & Nakano (1993) have presented evidence that the former can be described as a multiplicative cascade of the enstrophy dissipation, but the most obvious phenomenon of two-dimensional flows is that the vorticity condenses into a finite number of large coherent vortices (McWilliams 1990*a*), after which the self-similarity of the decay is either destroyed (Santangelo, Benzi & Legras 1989) or, at least, satisfies very different laws (Carnevale *et al.* 1991).

It is easy to interpret this result in the light of the general blocking effect discussed here. Vorticity is conserved in two-dimensional inviscid flows. If there is an inertial cascade, it can only act by separating pre-existing higher from lower vorticity into different patches, but the overall mean does not change until viscosity begins to act. Note that the same is true of all the moments of the vorticity field, and that the same argument could be made about the square of the vorticity, or about its absolute magnitude, thus avoiding problems with signed quantities. In a scenario similar to the one described for three-dimensional turbulence, large patches of vorticity cross the coherence boundary at the initial stages of their evolution. A given vortex patch is destined to be coherent or not from the start.

The result is that two-dimensional turbulence can be considered a mixture of two 'fluids'. Low-vorticity fluid is destined to cascade to small scales, where viscosity damps it, while high-vorticity patches decouple from that evolution and interact with each other according to different laws. The behaviour of both components is very different. Since coherent vorticity tends to form roughly circular vortices which are very stable in two dimensions, the low-vorticity component decays before the coherent eddies have time to interact with one another. During most of the lifetime of decaying two-dimensional turbulence, only the blocked vortices persist. That they do not participate in the overall enstrophy cascade to higher wavenumbers was shown by McWilliams (1990*b*), and their coherent evolution has been described, for example, by Carnevale *et al.* (1991). Borue (1993) has shown that, in continuously forced two-dimensional turbulence at sufficiently high Reynolds numbers, both processes coexist.

The term 'blocking' has historically been used in meteorology with a somewhat related meaning. It describes the persistence of very stable large-scale structures, often linked to topography, which impede the normal evolution of the weather. Flierl (1987) reviewed this phenomenon, including examples such as the Great Red Spot on Jupiter and stable weather systems over the Earth's oceans. Although very important in meteorology, this concept has been largely superseded in turbulence theory by the later realization that the formation of large-scale structures is a natural outcome of the two-dimensional inverse energy cascade.

### 3.3. *A model problem*

The essence of the two previous cases is that self-similarity is broken. The basic assumption up to now has been that the probability distribution of $u_{n+1}$ is only a function of $u_{n+1}/u_n$. This follows from the absence of an external scale for $u_n$, and is valid as long as all the $u$ are strictly independent of one another. But, as soon as we interpret the cascade as an extended system, new possibilities arise even in the absence of external scales. We can for example have

$$p_t(u_{n+1}|u_n) = u_n^{-1} w[u_{n+1}/u_n;\, u_n/S_n(1)], \tag{3.7}$$

where the mean value over the whole ensemble acts as an internal scale. This changes the expected result. If, for example, we assume that the variance of the multiplicative factors decreases as $u/S(1)$ increases, we might expect a relatively stable population of strong fluctuations to be established, and the generation of very strong tails to be inhibited.

In figure 4 we show the result of one such experiment, where we have used a modified version of the binomial process discussed in § 3.1. At each step the variables are multiplied by either $1+\alpha$ or $1-\alpha$ with equal probability, where

$$\alpha = 0.21 \left(1 + \frac{u}{5S(1)}\right)^{-m}. \tag{3.8}$$

When the exponent is chosen as $m = 0$ we recover the self-similar binomial process discussed before, but high exponents induce strong blocking of large perturbations. The result agrees with the intuitive arguments. High blocking leads to a bimodal distribution of 'coherent' and 'incoherent' objects, which develops independently of whether the initial conditions are concentrated or not.

## 4. Inverse cascades

We have seen that local 'direct' cascades, those in which the probability distribution of an element depends only on the state of its unique predecessor, lead naturally to intermittency and power laws. Novikov (1971) remarked that exact power laws can only be expected if the cascade is local in that sense, and if, in addition, the factors are statistically independent of one another. We have seen that neither of those conditions is strictly satisfied in most systems, even if only because of the blocking effects mentioned in the previous section, but little is known about the non-local

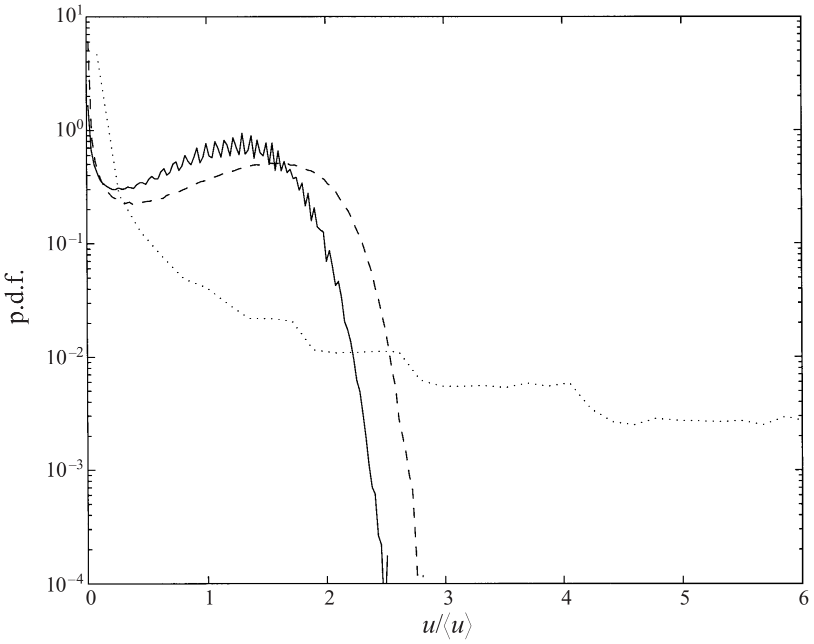


FIGURE 4. P.d.f. of $u$ for the nonlinear multiplicative cascades described by (3.7)–(3.8). All the results are shown after 200 steps, on a universe of $9 \times 10^5$ points. ········, $m = 0$, with initial conditions uniformly distributed in (0, 1); – – – –, $m = 9$, same initial conditions; ———, $m = 9$, initial conditions uniformly distributed in (0.9, 1).

case. A physically interesting example is the inverse cascades, in which a quantity is transferred from smaller to larger scales.

We have already mentioned the inverse energy cascade in two-dimensional turbulence. Other processes that can be interpreted in this way are the transfer of shear stress (momentum) away from the wall in pipes and boundary layers (Jiménez 1999), and the growth of the largest structures in free shear flows, described by Brown & Roshko (1974). In all these cases large eddies receive information from several smaller ones and, even if a multiplicative process may still be a good model due to scale similarity, the multiplication is applied to a function of all the predecessors within some local neighbourhood.

It is clear that intermittency is less likely in this case than in the ones discussed up to now. Consider the extreme example in which, before multiplication, an average is taken at each cascade step over the whole flow field. All the statistical variability would then be due to the generating factors, and the flatnesses would not grow as the number of steps increases.

It is in fact true that some inverse cascades are less intermittent than direct ones. Smith & Yakhot (1993) report simulations in which two-dimensional turbulence in a periodic box is forced at the small scales and allowed to cascade to larger ones. Until the energy reaches the box size the structure functions stay slightly sub-Gaussian, with no indication of intermittency. It is interesting that, once the energy hits the box, coherent structures form, and that they were shown by Borue (1994) to dominate the flow at large times. A recent experiment by Paret & Tabeling (1998) reinforces this observation. Two-dimensional turbulence is forced at small scales in a shallow basin of conducting fluid, and damped by the friction with the bottom which, since it is proportional to the velocity, acts predominantly on the larger scales. When the friction is adjusted to be strong enough that the largest scales are damped before they reach the box size, a steady inverse cascade is established which is not intermittent. If the friction is weaker, and the cascade reaches the box size, coherent structures form which control the velocity field. The two observations suggest that the inverse cascade itself is not intermittent, and that it is only when it is disturbed by finite-size effects that structures have time to form.

Other examples are harder to interpret. The largest scales of boundary layers are intermittent, in the sense that laminar regions alternate with turbulent ones (Hinze 1975), and that the flatness of the velocity increases as its moves 'down' the cascade and away from the wall. Perhaps it is also true in this case that the inverse cascade becomes intermittent only when it reaches scales of the order of the boundary layer thickness.

Experiments with simple numerical cascades display a broad range of behaviours. Consider the four cases in figure 5. The first experiment is a local multiplicative cascade generated by repeatedly multiplying a vector of 150 000 numbers by factors uniformly distributed in (0, 1). The random multipliers are regenerated at each step. The fourth-order flatness is given as the dotted line in figure 5($c$), and increases exponentially with the step number, as predicted by (2.9). For uniformly distributed factors, $F_w(\lambda) = 2^\lambda/(\lambda + 1)$. The resulting power law has been included in the figure and agrees well with the data.

It is also easy to understand the second experiment, represented in the figure as a dashed line. The generating factors are the same as before but, before applying them at each step, the vector of cascading numbers is randomly shuffled and convolved with a box filter of width five. That is, a running average is taken over each five neighbouring elements. It is seen in the figure that the flatness in this case reaches a steady value. This is a less extreme case of the averaging procedure mentioned at the beginning of this section. Even averaging over only five samples reduces the distribution at each step to approximately Gaussian. The observed distribution, which is shown in the figure after the multiplication step, is therefore never more than one step away from being Gaussian. Under the assumption that the distribution after the local averaging is exactly Gaussian, it is possible to write an evolution equation for the flatness. It approaches exponentially a fixed point which depends on the details of the process and which, in this case, is $F(4) = 608/121$. It has been included in the figure as a short segment on the right-hand side, and agrees excellently with the data.

Shuffling at each step is important, because it destroys the spatial structure. In the third experiment the multiplication and the filtering are the same as in the second one, but the shuffling is not done. It is useful in the next two cases to consider the cascading vector as a one-dimensional discrete function of its index, $u_j$, and the averaging as a convolution filter. The data can then be described by their Fourier transform,

$$u_j = \int_{-\pi}^{\pi} \hat{u}(\kappa) \exp(\mathrm{i}\kappa j)\, \mathrm{d}\kappa, \tag{4.1}$$

and characterized by its power spectrum $E(\kappa) = |\hat{u}^2|$. Because of the discrete nature of the data, only wavenumbers up to $\kappa = \pi$ are relevant. The filter is then described by its transfer function, and the power spectrum of the data after filtering is $T(\kappa)E(\kappa)$. The transfer functions of the different filters used in these experiments are given in figure 5($a$), and the power spectra of the resulting data sets in figure 5($b$).

In the absence of any filtering, the transfer function is flat, and so is the spectrum

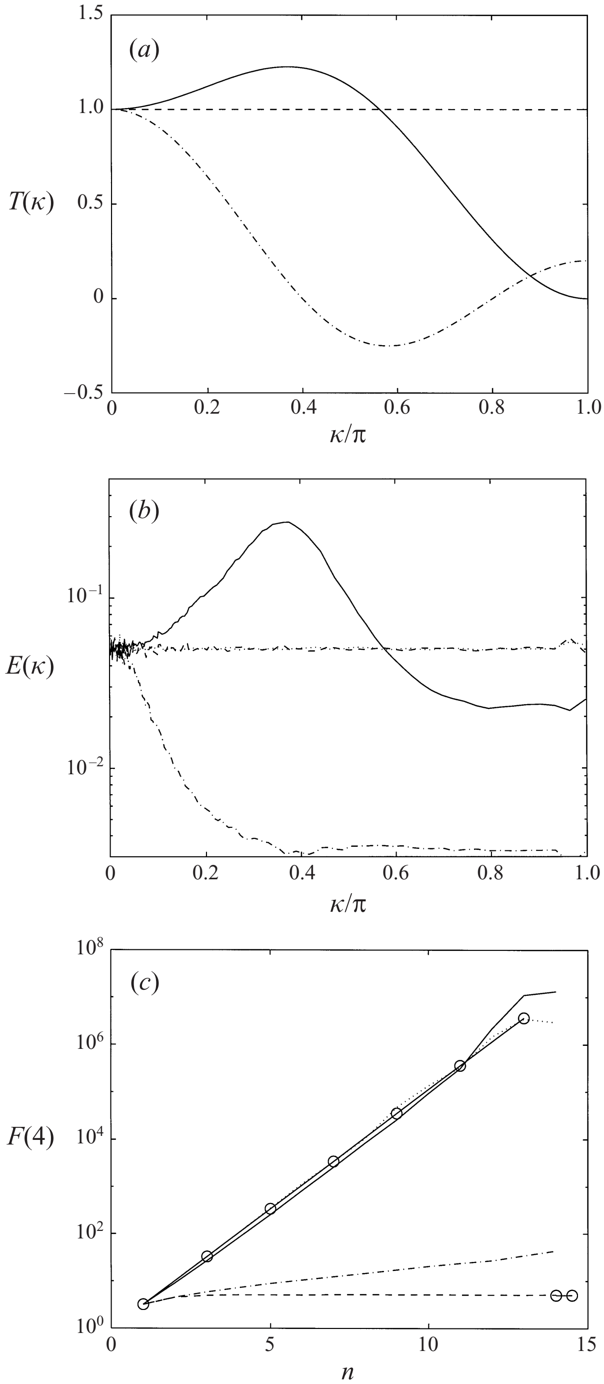


FIGURE 5. Non-local cascades described in the text. (*a*) Dispersion relation of the smoothing filters. (*b*) Power spectra after 20 cascade steps. (*c*) Fourth-order flatness, $S_n(4)/S_n(1)^4$, as a function of the cascade step: ········, no filter; – – – –, box filter, shuffled; —·—, box filter, unshuffled; ———, unstable filter; —○—, theoretical estimates.

of the data, corresponding to spatially uncorrelated white noise. The same is true in the presence of shuffling, where any spatial structure induced by the local averaging is destroyed by the repeated randomizations.

The effect of the box filter is to damp the higher wavenumbers, so that the data tend to develop short-range spatial correlations, and the spectrum becomes concentrated in the long wavelengths. In some sense the data behave as if they belonged to a coarser vector in which neighbouring elements evolve together, multiplied by factors which are averaged over a few neighbouring points, and which are therefore less random than the actual ones. The flatness increases exponentially, because self-similarity is preserved, but the slope is much lower than in the unfiltered case.

That the slower growth of intermittency is due to the local correlations developed by the data in the unshuffled experiment, rather than to the averaging of the factors at each step, is shown by the last experiment in this series. We apply a filter with the same width of five points, but tailored so that it has an unstable range of wavenumbers, amplifying wavelengths of three or four sample points. This destroys the coherence. Neighbouring elements are again decorrelated, and the spectrum develops a peak over the range of unstable wavenumbers. All the high-order flatnesses follow again a power law, identical to that of the unfiltered multiplicative process.

The reasons for these different behaviours are not well understood. It is easy to find filtering strategies that result in power laws intermediate between those of the unfiltered and of the fully averaged experiments, and there is even some indication that very unstable filters lead to super-exponential growth of the moments.

## 5. Kinematics versus dynamics

It should finally be made clear that the multiplicative model discussed up to now is in most cases a purely kinematic description, whose relation to the actual dynamics of the system is uncertain. Consider for example the case of three-dimensional turbulence. Most analyses consider the distribution of the integrated dissipation over segments of a one-dimensional section of the flow field

$$\overline{\varepsilon}_\ell(X) = \ell^{-1} \int_X^{X+\ell} (\partial u/\partial x)^2 \,\mathrm{d}x. \tag{5.1}$$

Because an integral is, by definition, conservative, and because the integrand is non-negative, the only multiplicative cascades considered are usually conservative ones, with factors in $(0, \ell_1/\ell_2)$. While that is obviously correct for the '*a posteriori*' kinematic description, it would be wrong to conclude that the energy cascade is locally conservative.

Consider a decaying eddy. The energy transfer among scales can be quantified by considering the flow field $\overline{u}_\ell$, smoothed by some appropriate local filter of width $\ell$, such as was introduced for large-eddy simulations by Leonard (1974). The velocity is separated into a large-scale component $\overline{u}_\ell$, and a sub-grid fluctuation $u'_\ell$. An energy equation can be written for the kinetic energy of the large-scale velocity, which has a sink term, representing the energy dissipated to the sub-grid scales, and the divergence of a spatial flux by which eddies interchange energy with each other. For control volumes of size $\ell$, the integrals of both contributions are $O(u'^3_\ell \ell^2)$. The internal spatial fluxes tend to cancel over larger volumes, and only the down-scale energy transfer rate is left on the average. The large volumes, however, also interchange energy with their large neighbours, and those fluxes are of the same order as their integrated dissipation. It is only in the average that the energy transfer rate has to be equal to the dissipation, and the former is never locally conserved within the inertial range. The spatial fluxes are the reason why the dissipation is unequally distributed when an eddy decays, since the two halves interchange energy which is of the same order as what they receive. Their presence also implies that the dissipation of a given eddy is not conserved under cascading, since part of it may be exchanged with its immediate neighbours. In addition, part of the energy flux is actually due to the pressure fluctuations, which are not local, and can redistribute energy over large distances.

The real dynamics of the cascade does not therefore have to be conservative, and the multiplicative factors are not necessarily bounded. It has in fact been found numerically that the local energy transfer rate in the inertial range can be negative, towards larger scales, in about 40% of the points, and that the overall transfer to small scales is the difference between this reverse component and the slightly larger down-scale flux over the rest of the volume (Piomelli *et al.* 1991). None of this is apparent from the *a posteriori* analysis of the quantity (5.1).

The multiplicative factors of the dynamical generating process can be very different from those of the kinematic analysis. Consider the trivial, and extreme, example of a one-dimensional equivalent of (3.1)–(3.2), in which a segment is bisected in each cascade step, and the dissipation is randomly weighted with factors $\alpha$ and $2-\alpha$. The cascade conserves the mean dissipation, and it is easy to see that the dynamic scaling exponents should be

$$\sigma_\lambda = -\log_2\left[\frac{\alpha^\lambda + (2-a)^\lambda}{2}\right]. \tag{5.2}$$

This means that the moments of the dissipation $\varepsilon_\ell$, which is what eddies of size $\ell$ have before they break down, behave as $\ell^{\sigma_\lambda}$.

Assume now that the redistribution of the dissipation is done by random exchange among the eddies, which we will mimic by shuffling the segments randomly after each cascade step. All the spatial structure is lost in this way, but the cascade does not change, since we have assumed it to be strictly local. Assume next that the scaling exponents are measured as in the experiments. The final dissipation, after $N$ steps, is broken into blocks of $\ell$ elements, and the scaling analysis is done on the structure functions of the integrated variables

$$\bar{\varepsilon}_\ell(K) = \ell^{-1}\sum_{K}^{K+\ell}\varepsilon_j. \tag{5.3}$$

Because $\bar{\varepsilon}$ is the sum of a large number of random numbers, its distribution quickly becomes Gaussian, and its structure functions behave classically, $\sigma_\lambda \approx 0$. In numerical experiments with $N = 12$–$16$, the Gaussian behaviour is recovered for blocks which are 4–5 cascade steps above the final 'Kolmogorov' scale.

The complete shuffling of the previous example is not representative of the situation in turbulence, where some spatial structure is obviously left in the dissipation. But some amount of redistribution is present in real flows, and more realistic models are also found to have a measurable effect on the scaling exponents. Consider the following milder version of the shuffled bisection process. In a complete cascade step each segment is broken in two, each of which is multiplied randomly by $\alpha$ or $2-\alpha$. But before the next bisection is done, each element swaps its two sub-segments with its two nearest neighbours. This swapping mimics more closely than in the previous example the energy fluxes in real flows, and spatial structure is not completely destroyed.

Numerical experimentation reveals however that the *a posteriori* kinematic scaling exponents are reduced with respect to (5.2) by about a factor of two.

While none of these examples can claim to model turbulent flows in any real sense, they illustrate that simple effects can substantially modify the relation between kinematically measured scaling exponents and their underlying dynamics. Another example of the influence of spatial relationships in multiplicative cascades is discussed by Greiner, Eggers & Lipa (1997).

Some progress has been made recently on the study of intermittency independently of heuristic multiplicative models. Beside the structural observations mentioned in the previous section, the model problem of a passive scalar advected by a random velocity field has been largely solved, and intermittent scaling exponents computed from first principles. The reader should consult the papers by Kraichnan (1994), Gawedzki & Kupianen (1995) and Benzi, Biferale & Wirth (1997).

## 6. Summary and conclusions

We have summarized some aspects of the appearance of intermittency in cascading processes, with emphasis on its application to Navier–Stokes turbulence. We have shown that the formation of small scales is essentially independent of the generation of intermittency. The latter is basically the result of a multiplicative cascade, which is a consequence of locality and of the absence of characteristic intensity scales. Its main effect is the generation of strong events, whose magnitudes are much higher than the mean level of fluctuations in the system. We have mentioned in passing that such processes may be important in other systems besides turbulence, or even besides physics, and have given references to their application to the economic and social sciences.

The theory is best developed for local direct cascades, which are essentially multiplicative Markov processes, and which lead directly to multifractals in their geometric interpretation. We show in the Appendix the relation between the probability distributions found in those processes and the lognormal. Both may agree over long ranges, spanning in some cases many orders of magnitude, but the moments obtained from the lognormal are in general wrong, even for low orders.

We have highlighted some of the limitations of these simple local models. The first is the formation of coherent structures, which are an intrinsic consequence of the generation of very strong fluctuations. In essence the most intermittent components of the system decouple from the background and evolve as if they were in the absence of environmental perturbations. The average value of the perturbations acts then as an extra intensity scale and, since the p.d.f.s of multiplicative cascades do not tend to steady states, the detailed self-similarity of the solution is broken for elements which are very far from the mean. We have argued that the coherent small-scale vortices of three-dimensional turbulence, and the persistent eddies of two-dimensional flows, are examples of this phenomenon. The theoretical treatment of such nonlinear multiplicative cascades is in its infancy.

We have briefly discussed the effect of spatial non-locality, the best known examples of which are the inverse cascades of turbulent flows. In those cases the evolution of an element depends on some average over several of its predecessors. An elementary argument suggest that such cascades should be less intermittent than strictly local ones, and there is some experimental evidence to support that conclusion, but simple numerical experiments show that this is not necessarily the case, and that it is possible to have non-local cascades which are less, as much, and possibly more, intermittent than local ones based on the same random multipliers. The difficulty here is not the same as in the previous case, since it is not due to nonlinearity but to the introduction of a new (spatial) dimension.

We have finally noted that the kinematic description of cascades in terms of the statistics of their finest scales may lose some of the information on the dynamics of the process that originated it, and that the results of both descriptions might be quite different.

Random, scale-invariant, systems appear in many areas of science and engineering, and underlie the behaviour of many complex systems. Besides their obvious applications to fluid turbulence, one can think of the economy, politics and even of the origin of life. In all these cases the emergence of uncommon individuals, with properties substantially different from the average, is an interesting occurrence, and multiplicative cascades may be a useful description of their evolution. In the present theory of local cascades, and in the corresponding description of self-similar multifractals, we have the equivalent of the theory of linear ordinary differential equations. In this article we have tried to point to the need of extending it, first to the equivalent of nonlinear and then of partial differential equations.

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## Appendix. The lognormal approximation

Consider the product of $n$ independent positive random variables $u_j$, whose logarithms $z_j = \log u_j$ have a common probability distribution. Assume also that $u_j$ is normalized so that the mean value $\langle z_j \rangle = 0$. Consider

$$U = \prod_{j=1}^{n} u_j, \qquad Z = \sum_{j=1}^{n} z_j. \tag{A 1}$$

It follows from the central limit theorem that, if $z$ has a finite variance, the distribution of $Z$ approaches a Gaussian with zero mean value and variance, $\sigma_n = n^{1/2}\sigma$, where $\sigma$ is the variance of the $z_j$. This approximation holds, for distributions with a non-zero third moment, in a neighbourhood of the origin of the order of (Feller 1971, p. 533)

$$|Z| \leqslant O(n^{1/6}\sigma_n) = O(n^{2/3}\sigma). \tag{A 2}$$

Within these limits the distribution of $U$ is approximately lognormal,

$$p_{LN}(U) \approx \frac{1}{\sqrt{2\pi n}\,\sigma U} \exp\left[-\frac{\log^2 U}{2n\sigma^2}\right], \tag{A 3}$$

but it is easy to show that all the 'important' values of $U$ fall outside the domain of validity of that approximation. It follows from the discussion of multiplicative processes in § 2 that the mean value of $U$ is $\chi^n$, where $\chi = \langle u \rangle$ which, since the graph of the exponential is concave upwards, is

$$\chi = \langle u \rangle = \langle e^z \rangle \geqslant \langle z \rangle = 0. \tag{A 4}$$

Equality only holds for distributions concentrated at one point. The situation after $n \gg 1$ cascade steps is that the distribution for $Z$ is approximately Gaussian within a

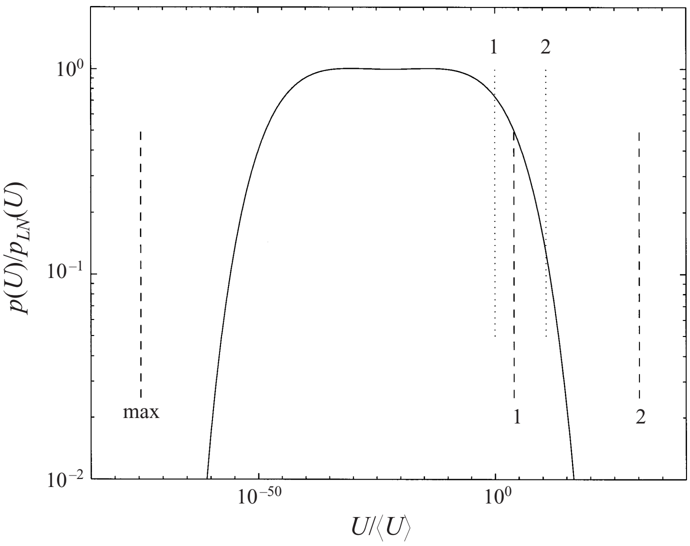


FIGURE 6. Ratio between the exact p.d.f. of the binomial process (A 5) and its lognormal approximation. The dashed line marked 'max' is the position of the maximum of the lognormal. Those marked '1' are $S(1)$, and those marked '2' are $S(2)^{1/2}$. ········, Moments computed from the true distribution; - - - -, moments computed from the lognormal. $\gamma = 3$, $N = 100$.

segment of length $O(n^{2/3})$, but the 'important' mean value $\log\langle U\rangle = O(n)$ lies outside that segment. The maximum of the lognormal distribution (A 3) is at $U = \exp(-\sigma_n^2)$, which is also outside the range of validity of the approximation.

When the p.d.f. is plotted in terms of $U/\langle U\rangle$, the segment in which the approximation applies is below $U/\langle U\rangle = O(\chi^{-n})$, but it may span many orders of magnitude both in the independent variable and in the p.d.f., $\Delta \log U \approx -\Delta \log p = O(n^{2/3})$. Moreover since the process of deforming the independent variable does not change the probability measure, the mass contained in this exponentially short segment is the same as that within $O(n^{1/6})$ standard deviations of the approximately Gaussian distribution of $z$, and corresponds asymptotically to the full mass of the distribution. The approximation is nevertheless misleading, since it does not determine any of the moments of the true distribution, which are controlled by the very stretched $O(1)$ tail which it neglects.

Consider for example the bisection process defined by the transition probability

$$w(x) = \tfrac{1}{2}[\delta(x-\gamma) + \delta(x-\gamma^{-1})]. \tag{A 5}$$

which leads after $N$ steps to the binomial density

$$U(k) = \gamma^{2k-N}, \qquad p(U) = \frac{1}{2U_k \log\gamma}\,\frac{N!}{2^N k!(N-k)!}. \tag{A 6}$$

The variance of the lognormal approximation is

$$\sigma_n = n \log^2 \gamma. \tag{A 7}$$

The ratio between the true and lognormal densities is shown in figure 6. In the particular case used in the figure the approximation holds over more than 30 orders of magnitude both in $U$ and in the p.d.f., in spite of which the mean, the standard deviation, and the maximum of the lognormal, are outside its range of validity. If the lognormal model had been used to compute the mean value of the distribution, the error would have been more than five orders of magnitude, while the error in $S(2)^{1/2}$ would have been of the order of $10^{20}$.

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