SPECTRAL ELEMENT STABILITY ANALYSIS OF VORTICAL FLOWS

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Abstract The scope of the present study is to demonstrate the use of spectral/*hp*-element methods in understanding the global instability mechanisms of vortex dominated flows. Using a BiGlobal stability analysis, analytically constructed and numerically evaluated base flows have been investigated, with the leading eigenvalues obtained by the Arnoldi algorithm. Subsequently, Direct Numerical Simulation (DNS) was used to investigate the non-linear development of an unstable Batchelor vortex. It was found that a spiral-type instability, if allowed to develop in an axially unconstrained manner, leads to an axial loss of energy and the formation of a stagnation point.

Keywords: BiGlobal; stability; vortex breakdown; DNS; PSE

1. Introduction

Although vortex breakdown has been researched for some time, there remains no accepted explanation of the phenomenon. Following Leibovich, 1978, breakdown is defined as, ‘a disturbance characterised by the formation of an internal stagnation point on the vortex axis, followed by reversed flow in a region of limited extent’. The main theories associated with breakdown are those of vortex stability; and the wave-motion theories, primarily attributed to Squire (1960) and Benjamin (1962).

The concept of hydrodynamic stability, and its advancement to global linear instability theory (summarised by Theofilis, 2003), has resulted in considerable investigation into the unstable modes of the Batchelor (1964) vortex model. Initially investigated by Lessen et al. (1974) and more recently by Ash and Khorrami (1995), several spiral-type modes of instability exist. It is not immediately evident, however, how an instability can lead to an abrupt change in flow structure; although the DNS of Abid and Brachet (1998) does relate the non-linear development with a lateral expansion. The aim of the current research is to unify linear stability analysis with three-dimensional DNS to show how a slowly developing spiral instability can lead to an axial stagnation.

2. Numerical Method

The numerical method applied consists of solving the eigenvalues of a matrix system corresponding to the linearised incompressible Navier-Stokes equations. Following the methods of Barkley and Tuckerman (2000), an exponential power method – coupled with an Arnoldi algorithm – is used to evaluate the leading eigenvalues of the system; which is reduced to a Krylov subspace spanning the number of eigenvalues sought. Validation of the method was achieved by comparing the results for an isolated Batchelor Trailing Vortex (BTV) with the classical one-dimensional stability analysis of Mayer and Powell (1992), which assumes both a streamwise and an azimuthal Fourier decomposition.

Linearised stability analysis is based upon the decomposition of all flow variables into a steady mean component upon which small-amplitude three-dimensional disturbances are permitted to develop (i.e. $\mathbf{q} = \bar{\mathbf{q}} + \mathbf{q}'$). By allowing a mild dependence of the base flow on the streamwise spatial coordinate z, an eigenmode Ansatz is introduced, according to which

$$\mathbf{q}'(x, y, z) = \hat{\mathbf{q}}(x, y, z^*) \exp i\Theta + c.c. \quad (1)$$

$$\Theta = \Theta_{3D} = \int_{z_0}^{z} \beta(\xi) d\xi - \Omega t \quad (2)$$

Applied to the linearised Navier-Stokes equations this leads to the following system of equations that define the Parabolised Stability Equation (PSE) concept (originally developed by Herbert, 1997) for three-dimensional flows

$$\hat{u}_x + \hat{v}_y + i\beta\hat{w} = -\hat{w}_z \quad (3)$$

$$\{\mathcal{L} - \bar{u}_x\}\hat{u} - \bar{u}_y\hat{v} - \hat{p}_x + i\Omega\hat{u} = \bar{w}\hat{u}_z + \bar{u}_z\hat{w} - \frac{2i\beta}{Re}\hat{u}_z \quad (4)$$

$$-\bar{v}_x\hat{u} + \{\mathcal{L} - \bar{v}_y\}\hat{v} - \hat{p}_y + i\Omega\hat{v} = \bar{w}\hat{v}_z + \bar{v}_z\hat{w} - \frac{2i\beta}{Re}\hat{v}_z \quad (5)$$

$$-\bar{w}_x\hat{u} - \bar{w}_y\hat{v} + \mathcal{L}\hat{w} - i\beta\hat{p} + i\Omega\hat{w} = \bar{w}\hat{w}_z + \bar{w}_z\hat{w} - \frac{2i\beta}{Re}\hat{w}_z + \hat{p}_z \quad (6)$$

Where $\mathcal{L} = (1/Re)\{\partial_{xx} + \partial_{yy} - \beta^2\} - \bar{u}\partial_x - \bar{v}\partial_y - i\beta\bar{w}$. Implicit in this derivation is that the disturbance takes the form of a rapidly varying phase function and a slowly varying shape function, for which second derivatives with respect to z (along with products of first derivatives) can be neglected.

3. Stability of a Batchelor Vortex

A single Batchelor vortex (defined by Batchelor, 1964) with a swirl value of $q = 0.8$ and a co-flow parameter of $a = 0$ has been investigated. A typical linearly unstable perturbation mode is illustrated in Figure 1 for a Reynolds

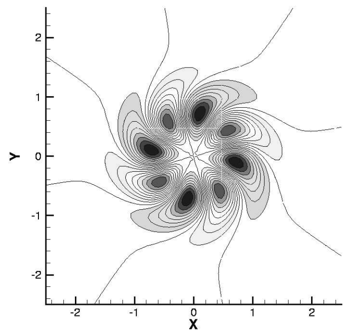

Figure 1. Two-dimensional view of the linear perturbation mode for an isolated Batchelor vortex with $\beta = 2.0$. Visualised using contours of axial velocity.

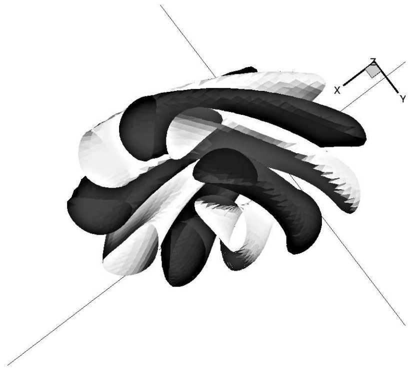

Figure 2. Three-dimensional view of the linear perturbation mode for an isolated Batchelor vortex with $\beta = 2.0$. Visualised using iso-surfaces of axial velocity.

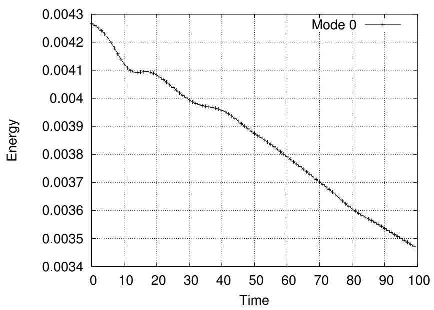

Figure 3. Temporal development of the kinetic energy within the zeroth axial Fourier mode.

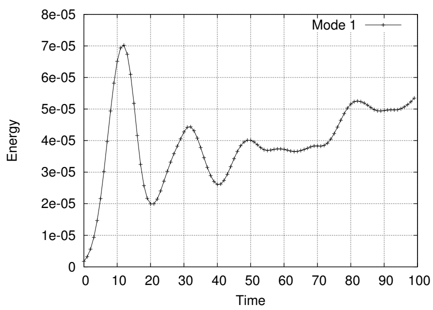

Figure 4. Temporal development of the kinetic energy within the first axial Fourier mode.

number based on the vortex core radius of $Re = 667$; the corresponding eigenvalue is $0.296 \pm 1.189i$. This was evaluated using a BiGlobal stability analysis; equivalent to Eqs. (3)–(6) with the RHS terms – which are related with derivatives of the basic flow and the disturbance terms in the z-direction – neglected.

4. Non-linear Development

The non-linear development of an isolated vortex has been analysed by DNS using $\mathcal{N}\epsilon\kappa\tau\alpha r$ [1], for $Re = 1000$. Initially, a periodic representation was applied in the axial direction, with the non-linear temporal development analysed from initial conditions constituting the isolated BTV with the first mode of instability superimposed as a small perturbation. As illustrated in Figures 3 and

[1] A spectral/*hp*-element solver developed by Sherwin and Karniadakis (1995)

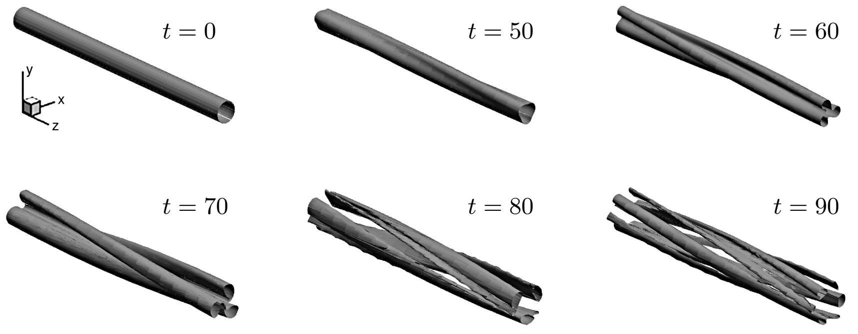

Figure 5. Non-linear development of an isolated Batchelor vortex assuming a periodic representation in the axial *z*-direction and a Fourier approximation consisting of the first 16 modes. Visualised using iso-surfaces of $\lambda_2 = -0.2$.

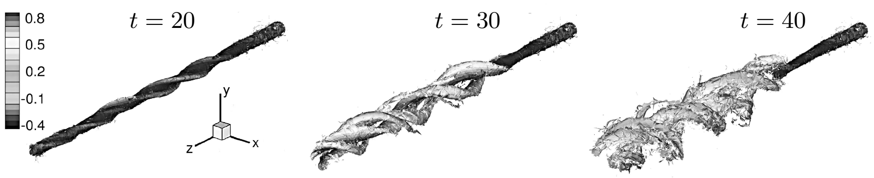

Figure 6. Three-dimensional temporal development of an isolated Batchelor vortex illustrating axial deceleration and formation of a stagnation point. Visualised using iso-surfaces of $\lambda_2 = -0.4$, shaded by axial velocity.

4, an energy transfer between the zeroth axial Fourier mode and the linear perturbation mode is identified. This is significant, since it implies that the growth of the linear instability leads to a loss of axial energy in the mean flow, which must be accompanied by a cross-section expansion to satisfy continuity, visualised in Figure 5. Consequently, the associated drop in axial velocity suggests a causal relationship between instability and breakdown.

Enforcing an axial periodicity in the solution restricts how the streamwise *w*-component of the velocity can change, limiting the extent of axial deceleration. To resolve this problem, 3D-DNS on the same BTV has been conducted (Figure 6). Although the initial development correlates well with the periodic representation, the axial deceleration now develops into a stagnation point – confirming the link between a spiral-type of instability and vortex breakdown.

5. Potential Applications

Whereas classical stability analysis places restrictions on the complexity of the instability modes that can be studied, BiGlobal analysis allows more gen-

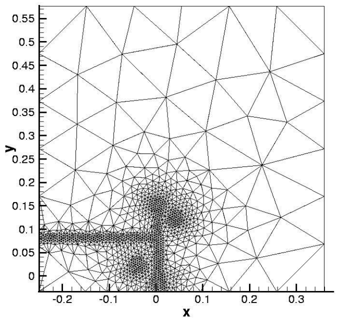

Figure 7. Computational grid in the wake of a low aspect ratio wing.

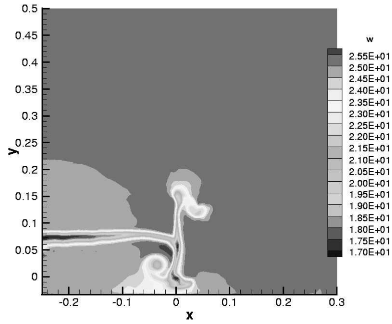

Figure 8. RANS-evaluated base flow computed about a low aspect ratio wing.

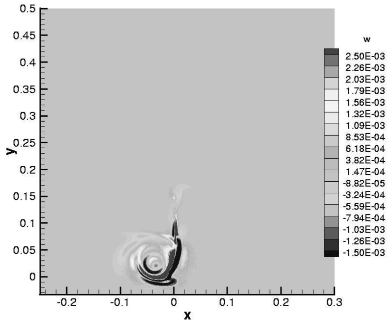

Figure 9. Short wavelength unstable perturbation mode of the trailing vortex.

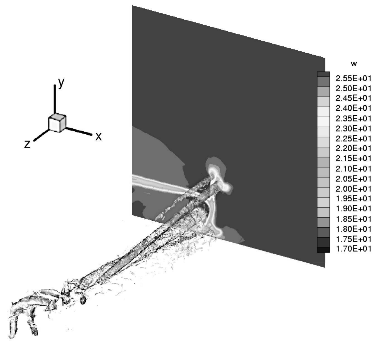

Figure 10. DNS of wake vortex system illustrating development of breakdown.

eral flows, with arbitrarily complicated vorticity distributions in the plane normal to the axial flow direction. The use of unstructured grids, coupled with the high spatial accuracy of spectral methods, also offers a significant flexibility to the method. For example, the complicated vortex system originating from a low aspect ratio wing close to the ground, along with a corresponding short-wavelength mode of instability, is illustrated in Figures 8 and 9 respectively; where the base flow was evaluated from a RANS-simulation.

Although there is a question on the appropriateness of taking such a solution from an analytic perspective, DNS of the RANS-evaluated wake system (visualised using λ_2 iso-surfaces in Figure 10) confirmed the development of an instability with a wavelength ($\beta = 100$) comparable with the most unstable perturbation mode. Analogous to an isolated Batchelor vortex, the development of the instability also leads to an axial loss of velocity in the vortex core. In a similar analysis, Crouch et al. (2004) used RANS-obtained basic states, coupled with a compressible BiGlobal stability analysis to obtain realistic eigenmodes related to the buffeting of an 18% thick bi-convex airfoil. This illustrates the scope of the method and suggests that it is valid.

6. Conclusions and Future Research

A unified approach to analysing vortex stability has been discussed, and a causal relationship between stability and breakdown implied. Spiral modes of instability were found to cause a lateral expansion of the cross-section, and a corresponding drop in axial velocity (a prerequisite of vortex breakdown). This confirms the proposals of Ash and Khorrami (1995), who describe breakdown as, 'a final outcome of vortex instability, with the caveat that vortex breakdown can also be produced by external means'.

External influences might include an adverse pressure gradient, which cannot be investigated through BiGlobal stability analysis. A suitable technique is the Parabolised Stability Equation concept derived in Section 2. This formulation permits flows with a mild variation in the axial direction and is currently being implemented to address the influence of axial pressure gradients, and their role in vortex instability and development to breakdown.

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