Intermittency in turbulence

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Abstract. Turbulence is characterized by the intermittent generation of very large gradients and velocity differences. The present understanding of this phenomenon is reviewed. In most cases the model of choice is the turbulent cascade, in which eddies become stronger as they decay to smaller sizes. The simplest model, a multiplicative process, has been studied in detail and leads to multifractals. It is shown that other processes are possible and that they fit the experiments better in some situations. Discrete long-lived structures form near the dissipative range of scales and may dominate the statistics if they are singular enough. That is the case in decaying two-dimensional turbulence. In the milder three-dimensional case, experiments favor a mixed multiplicative–stochastic model, which only tends to a multifractal in the limit of very large Reynolds numbers, probably beyond the reach of experiments.

Introduction

Intermittency has several meanings in turbulence, but the most interesting one for the purpose of this meeting is the tendency of the probability distributions of some quantities in three-dimensional Navier–Stokes turbulence, typically gradients or velocity differences, to develop long tails of very strong events. The problem is not that the distributions are not Gaussian. There is little reason in general for that to be true. The question is rather that the extreme tails become stronger as the Reynolds number increases, and that the trend does not show any sign of stopping at the highest experimental Reynolds numbers.

This is not altogether surprising, because turbulence is believed to be singular in the limit of infinite Reynolds numbers. In an equilibrium system, global energy conservation implies that the energy input should be equal to the average viscous dissipation per unit mass

$$\overline{\varepsilon}_v = \nu\overline{|\nabla \mathbf{u}|^2}, \tag{1}$$

where ν is the kinematic viscosity of the fluid, and $|\nabla \mathbf{u}|$ is the L_2 norm of the velocity gradient tensor. The average $\overline{()}$ is taken either over the whole flow or over a suitably designed ensemble of experiments. The basic experimental observation in turbulence is that the energy input required to maintain a turbulent flow becomes independent of the viscosity if the Reynolds number is high enough. In the limit of $\nu \to 0$, this implies that the velocity field has to develop infinite gradients.

The observation of intermittency goes beyond that singular behavior and implies that strong gradients become more common as the Reynolds number increases, even when measured in terms of their r.m.s. values. In particular, all the higher statistical flatness factors of the velocity gradients are thought to diverge in the inviscid limit.

Intermittency is not limited to gradients. Turbulence is a multiscale phenomenon, in which the ratio between the largest and the smallest length scales can reach $10^5 - 10^6$. The fundamental problem is to understand how large-scale quantities, such as the energy, are transported across that range of scales to the smallest eddies, where they can, for example, be dissipated. The generally accepted model is the cascade, introduced by *Richardson* [1920], which states that eddies of a given size only interact with those of somewhat larger or somewhat smaller sizes. Any interaction between eddies of very different sizes takes places through a sequence of such small cascade steps.

It is clear that, even if the underlying equations are deterministic, a phenomenon as complex as turbulence has some components that can best be described as random. In fact, when the physical consequences of the cascade were first explored by *Kolmogorov* [1941], he assumed that the process was complex enough for the eddies to lose all the memory of their previous histories, and that their properties after each cascade step could be described by purely random distributions. It then follows that there is an 'inertial' range of scales in which the eddies are too large for viscosity to be important, and too small to retain any effect of large-scale inhomogeneities. The Navier–Stokes equations are invariant to scaling transformations in that range, and the probability distributions of, for example, the velocity differences within an eddy, can only depend on the eddy size.

Consider for example the velocity difference Δu between two points separated by a distance r. The original Kol-

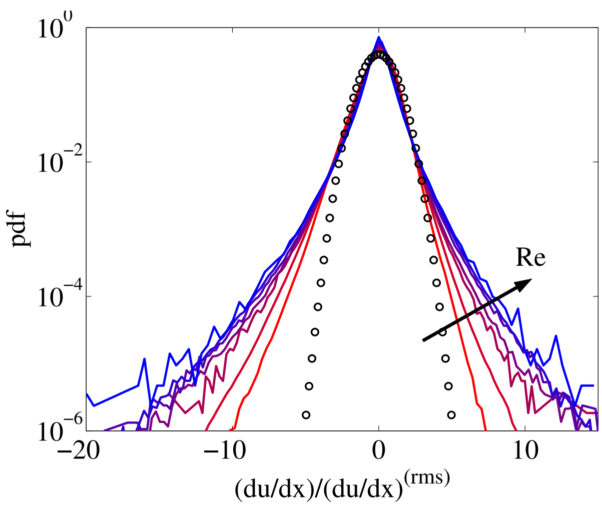

Figure 1. The p.d.f.s of the longitudinal velocity gradient for several Reynolds numbers, increasing in the direction of the arrow. Normalized with the standard deviation. $Re_L = 260 - 3.5 \times 10^6$. Symbols are Gaussian. Data from *Jiménez et al.* [1993]; *Belin et al.* [1997]; *Antonia and Pearson* [1999].

mogorov formulation assumes that the probability density function (p.d.f.) $p(\Delta u)$, is a universal function in the inertial range of scales, whose only parameter is a velocity scale depending on r. It then follows from energy conservation arguments that

$$p(\Delta u) = F\left[\Delta u/(\overline{\varepsilon} r)^{1/3}\right], \tag{2}$$

where $\overline{\varepsilon}$ is the average energy transfer rate across scales per unit mass. In an equilibrium system, it has to be equal to $\overline{\varepsilon}_\nu$.

Equation (2) is valid as long as the separation r is much larger than the Kolmogorov viscous cutoff $\eta = (\nu^3/\overline{\varepsilon})^{1/4}$, and much smaller than the integral scale of the largest eddies $L_\varepsilon = u'^3/\overline{\varepsilon}$, where u' is the root-mean-square value of the fluctuations of one velocity component. The extent of this inertial range is a function,

$$L_\varepsilon/\eta = Re_L^{3/4}, \tag{3}$$

of the Reynolds number $Re_L = u' L_\varepsilon/\nu$.

Intermittency in experiments

A consequence of the *strict similarity* hypothesis (2) is that the p.d.f.s of the velocity gradients, which are essentially the velocity differences across the Kolmogorov viscous scale η, should be universal. *Batchelor and Townsend* [1949] found that this was not true, and that the gradients become increasingly intermittent as the Reynolds number increases (Figure 1). The generation of intense gradients

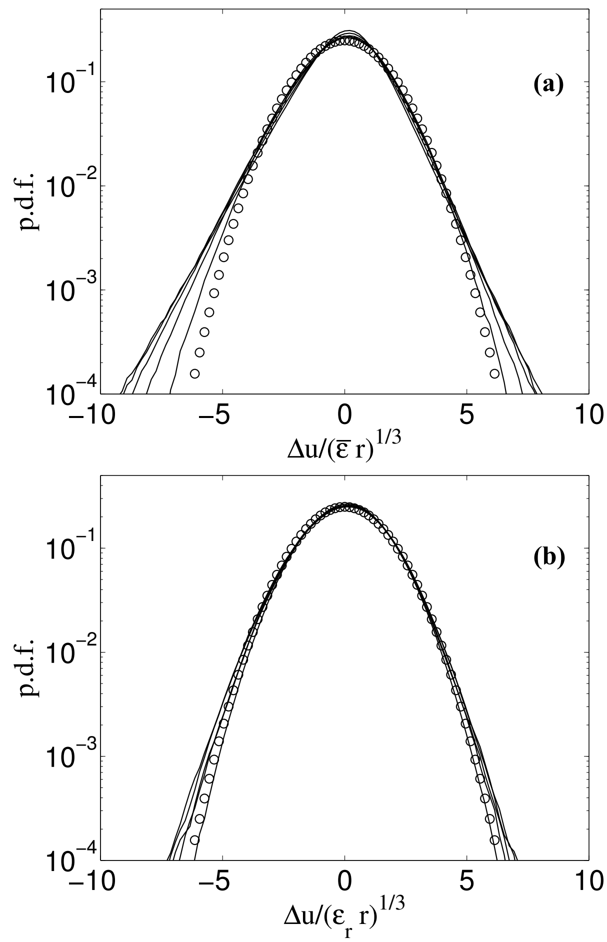

Figure 2. The p.d.f.s of the differences of the velocity component in the direction of the separation, for separations in the inertial range of scales. $r/L_\varepsilon = 0.02$ to 0.36, increasing by factors of 2; equivalent to $r/\eta = 180$ to 3000. Nominally isotropic turbulence at Reynolds number $Re_L = 10^5$. In the top figure Δu is normalized with the global energy dissipation rate $\overline{\varepsilon}$; distributions are wider as the separation decreases. In the bottom figure Δu is scaled with the locally averaged dissipation over the separation interval. Data courtesy of *Belin et al.* [1997].

was also found to develop gradually across the inertial cascade. The distribution of the velocity differences across distances of the order of the integral scale is approximately Gaussian, but it becomes increasingly non-Gaussian as the spatial separation is made much smaller than L_ε (Figure 2a). It was also soon noted that it was theoretically difficult to justify how a formula such as (2), representing the p.d.f. of a local property, could depend on a single global parameter. In a sense, a small interval r has 'no way to measure' the averaged dissipation $\overline{\varepsilon}$. *Kolmogorov* [1962] himself sought to bypass that difficulty by substituting (2) by a *refined similarity* hypothesis,

$$p(\Delta u) = F\left[\Delta u/(\varepsilon_r r)^{1/3}\right], \tag{4}$$

where ε_r is no longer a global average, but the mean value of the dissipation over a ball of radius of order r, centered at the mid point of the interval. This refined similarity is better satisfied by experiments (Figure 2b) although, from the practical point of view it just transfers the problem of characterizing the distributions of Δu to the characterization of the statistics of ε_r.

It has become customary to measure the behavior of $p(\Delta u)$ in terms of its structure functions,

$$S(n) = \int_{-\infty}^{\infty} \Delta u^n p(\Delta u)\, \mathrm{d}\Delta u, \tag{5}$$

which can be normalized as generalized flatness factors,

$$\sigma(n) = S(n)/S(2)^{n/2}. \tag{6}$$

It would follow from the strict similarity hypothesis (2) that

$$S(n) \sim r^{n/3} \tag{7}$$

and that all the $\sigma(n)$ should be independent of the separation. Figure 3 shows that this is not true. The flatness increases as the separation decreases, and it only levels off at lengths of the order of the Kolmogorov viscous scale. This is where the statistics of the gradients are defined. For separations in the viscous range, the flow is smooth, Taylor series expansions can be used, and $\Delta u \approx (\partial_x u) r$. It follows that in that range

$$\sigma(n) \approx \overline{(\partial_x u)^n} / \overline{(\partial_x u)^2}^{\,n/2}. \tag{8}$$

From Figure 3, the velocity gradients become increasingly non-Gaussian as the distance between L_ε and η grows with the Reynolds number.

The effect of structures

Because the velocity difference between two points which are not too close to each other can be expressed as the sum of velocity differences over subintervals, a loose application of the central limit theorem would suggest that its p.d.f. should be roughly Gaussian. The key conditions for that to happen are that the summands should be mutually independent, and that their probability distributions should have comparable finite variances [*Feller*, 1971, pages 169–172]. The first of those two conditions is probably a good approximation when the separation is much larger than the viscous cutoff, but the second one depends on the structure of the flow. The experimental non-Gaussianity suggests the presence of occasional very strong velocity jumps. In the viscous range of scales, those structures have been identified both experimentally and numerically as very strong linear vortices, in whose neighborhoods the strongest velocity gradients are generated [*Jiménez et al.*, 1993; *Belin et al.*, 1996]. An

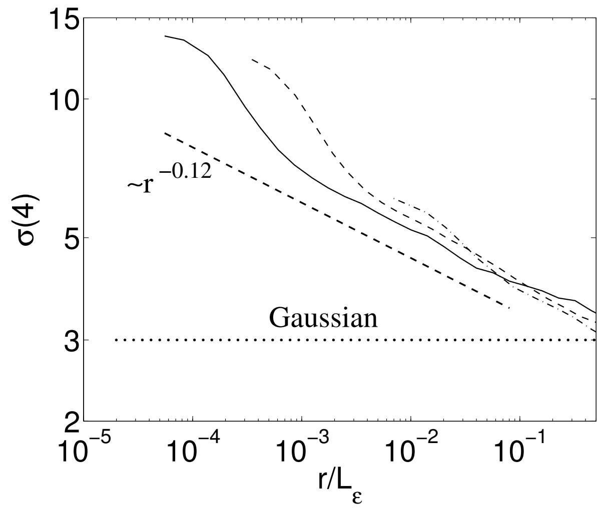

Figure 3. Fourth-order flatness of the differences of the velocity component in the direction of the separation, for separations in the inertial range of scales, $r/L_\varepsilon = 0.5$ to $r/\eta = 2$. The Reynolds numbers of the different flows range from $Re_L = 1800$ to 10^6. Data from *Belin et al.* [1997].

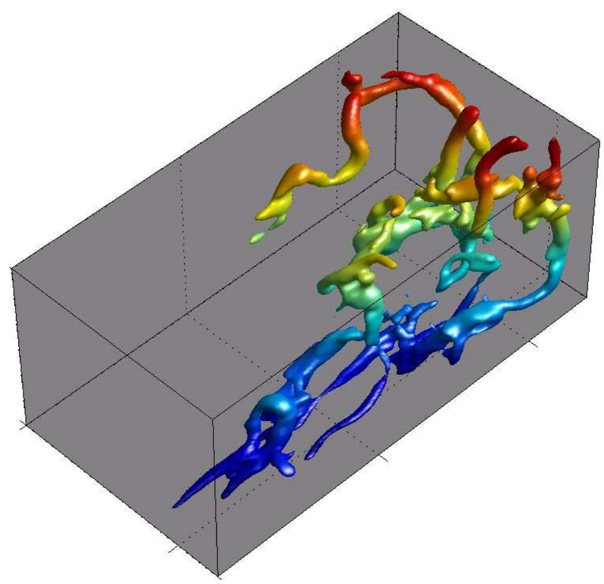

Figure 4. Intense vortex tangle in the logarithmic layer of a turbulent channel. The vortex diameters are of the order of 10η, and the size of the bounding box is of the order of the channel width. Data from *del Álamo et al.* [2006].

example of a tangle of such structures is shown in Figure 4.

In another example, the vorticity in decaying two-dimensional turbulence concentrates very quickly into relatively few strong compact vortices, which are stable except when they interact with each other [*McWilliams*, 1984]. The velocity field is dominated by them, and the flatness of the velocity increments reaches values of the order of

$\sigma(4) \approx 50-100$, even at moderate Reynolds numbers. That case is interesting because something can be said about the probability distribution of the velocity gradients [*Jiménez*, 1996]. We have noted that the p.d.f. of a sum of mutually comparable independent random variables with finite variances tends to Gaussian when the number of summands is large. This well-known theorem is a particular case of a more general result about sums of random variables whose incomplete second moments diverge as

$$\mu_2(s) = \int_{-s}^{s} x^2 p(x)\,\mathrm{d}x \sim s^{2-\alpha} \quad \text{when} \quad s \to \infty. \tag{9}$$

When $0 < \alpha \leq 2$, the sums of such variables tend to a family of *stable* distributions parameterized by α. The Gaussian case is the limit of that family when $\alpha = 2$. In the case of two-dimensional vortices with very small cores, the velocity gradients at a distance R from the center of the vortex behave as $1/R^2$. If we take s in (9) to be one of those velocity derivatives, its probability distribution is proportional to the area covered by gradients with a given magnitude, and

$$\mu_2(s) \sim \int_0^{s^{1/2}} R^{-4} 2\pi R\,\mathrm{d}R \sim s^{-1}. \tag{10}$$

The velocity derivatives at any point, which are the sums of the velocity derivatives induced by all the randomly distributed neighboring vortices, should therefore be distributed according to the stable distribution with $\alpha = 1$, which is Cauchy's [*Feller*, 1971, pages 574–581],

$$p(s) = \frac{c}{\pi(c^2 + s^2)}. \tag{11}$$

This distribution has no moments for $n > 1$. Its tails decay as s^{-2}, and the distribution of the gradients essentially reflects the properties of the closest vortex. In real two-dimensional turbulent flows, the distribution (11) is followed fairly well, but its extreme tails only reach to the maximum values of the velocity gradient found within the viscous vortex cores, which are not exactly point vortices.

The common feature of the two cases just described is the presence of strong structures that live for long times because viscosity stabilizes them. They are therefore more common than what could be expected on purely statistical grounds. They are responsible for the tails of the probability distributions of the velocity derivatives, but they are not the only intermittent features of turbulent flows. The increase of the flatness in figure 3 below $r \approx 50\eta$ is clearly connected with the presence of the coherent vortices, but even for larger separations there is a gradual increase of $\sigma(4)$. That suggests that the formation of intense structures takes place across the inertial range, but much less is known about those hypothetical inertial structures than about the viscous ones.

Cascade models

We can now recast the problem of intermittency in Navier-Stokes turbulence into geometric terms. We have already mentioned that the defining empirical observation for turbulence is that the energy dissipation given by (1) does not vanish even in the infinite Reynolds number limit in which $\nu \to 0$. This means that the flow has to become singular in the limit of infinite Reynolds number as $\overline{|\nabla \mathbf{u}|} L_\varepsilon/u' \sim Re_L^{1/2}$. The strict similarity approximation (2) assumes that those singularities are uniformly distributed across the flow, but the experimental evidence just discussed shows that this is not the case. The singularities are distributed inhomogeneously, and the inhomogeneity develops across the inertial cascade. The problem of intermittency is to characterize the geometry of the support of the flow singularities in the limit of infinite Reynolds number.

In the absence of detailed physical mechanisms for the dynamics of the inertial range, most intermittency models are based on plausible processes compatible with the invariances of the inviscid Euler equations. The precise power law given in (7) for the structure functions depends on the strict similarity hypothesis (2), but the fact that it is a power law only depends on the scaling invariances of the equations of motion. The energies and sizes of the eddies in the inertial range are too small for the integral scales of the flow to be relevant, and too large for the viscosity to be important. They therefore have no intrinsic velocity or length scales. Under those conditions, any function of the velocity which depends on a length has to be a power. Consider a quantity with dimensions of velocity, such as $u(r) = S(n)^{1/n}$, which is a function of a distance such as r. On dimensional grounds we should be able to write it as

$$u(r) = UF(\rho), \tag{12}$$

where $\rho = r/L$, and L and $U(L)$ are arbitrary length and velocity scales. The value of $u(r)$ should not depend on the choice of units, and we can differentiate (12) with respect to L to obtain

$$\partial_L u = (\mathrm{d}U/\mathrm{d}L)F(\rho) - U\rho L^{-1}(\mathrm{d}F/\mathrm{d}\rho) = 0, \tag{13}$$

which can only be satisfied if

$$\frac{\mathrm{d}F}{\mathrm{d}\rho} = cF \quad \Rightarrow \quad F \sim \rho^c, \tag{14}$$

where $c = L(\mathrm{d}U/\mathrm{d}L)/U$ is an undetermined constant. This suggests generalizing (7) to

$$S(n) \sim r^{\zeta(n)}, \tag{15}$$

where the exponents have to be empirically adjusted.

Only $\zeta(3) = 1$ can be derived directly from the Navier-Stokes equations. Equation (15) implies that the flatnesses

$\sigma(n)$ satisfy power laws with exponents $\zeta(n) - n\zeta(2)/2$. In Figure 3, for example, the fourth-order flatness follows a reasonably good power law outside the viscous range, consistent with $\zeta(4)-2\zeta(2) \approx -0.12$. The anomalous behavior near the viscous limit, and similar limitations at the largest scales, mean that only very high Reynolds number flows can be used to measure the scaling exponents, and that the range over which they are measured is never very large. Moreover, most of the mass of the integrand of the higher-order structure functions is in the extreme tails of the probability distributions of the velocity differences, which implies that very long experimental samples have to be used to accumulate enough statistics to measure the high-order exponents. For these and for other reasons the scaling exponents above $n \gtrsim 8 - 10$ are poorly known. This is unfortunate because we will see later that some of the most interesting intermittency properties of the velocity field, such as the nature of the flow singularities in the infinite Reynolds number limit, depend on the behavior of the $\zeta(n)$ for large n.

Another problem is that the cascades found in most experimental systems are short. The argument for (13) depends on the absence of natural velocity and length scales, which is never true. All flows have characteristic lengths and velocities, such as the integral parameters or those induced by viscosity. The invariance argument assumes that those scales are lost during the cascading process, which is a reasonable assumption after many cascade steps. But the reduction in length scale in a recursive cascade is exponential, and it does not take many steps to span the scale ranges found in nature. If we assume, for example, that the size of the eddies is reduced by a factor of two in each step, it would only take 20 steps to span the factor of 10^6 between the largest and the smallest length scales in even the highest-Reynolds-number geophysical flows.

Experimental values for the scaling exponents are given in Figure 5. They are generally smaller than the ones predicted by the strict similarity approximation, implying that the moments of the velocity differences decrease with the separation more slowly than they would if they were strictly self-similar, and suggesting that new stronger structures become important as the scale decreases.

Note that we have included in the figure values for odd-order powers. Up to now we have not specified which velocity component is being analyzed. Most experiments refer to the one in the direction of the separation, which is the easiest to measure, specially if time is used as a surrogate for distance. The longitudinal p.d.f.s are not symmetric, even in isotropic turbulence. Negative increments are more common than positive ones because of the extra energy required to stretch a vortex, and the effect is clearly visible in the distributions in Figures 1 and 2. The longitudinal odd-order structure functions do not vanish, and their scaling exponents are the ones used in the figure.

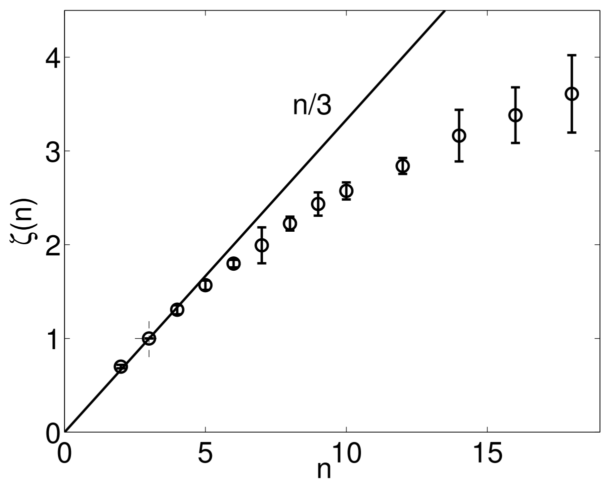

Figure 5. Longitudinal scaling exponents, from various experiments. Error bars are $\pm 2\sigma$ from the scatter among experiments.

Multiplicative models

In the absence of a general solution for the Navier–Stokes equations, most treatments of intermittency rely on phenomenological models, which are in reality little more than restatements of the experimental observations. The most successful ones have been those based on the concept of a multiplicative cascade, which is the next step in complication beyond the purely random *Kolmogorov* [1941] model. Consider some flow property v, such as the locally-averaged energy transfer rate by eddies of size r_k, as they cascade into smaller eddies of size r_{k+1}. Denote by $p_k(v_k)$ the probability distribution of the value of v at the step k of the cascade. The idea behind multiplicative cascades is that the process is still local in scale space, and stochastic, but that the intensity of the resultant eddies is not determined by a globally averaged property such as $\overline{\varepsilon}$, but by the intensity of the 'parent' eddy. We have already found that idea in the refined similarity hypothesis (4).

Assume that the cascade is Markovian in the sense that the probability distribution of v_k depends only on its value in the previous step,

$$p_{k+1}(v_{k+1}) = \int p_T(v_{k+1}|v_k;\, k)p_k(v_k)\, \mathrm{d}v_k. \tag{16}$$

This is in contrast to some more complicated functional dependence, such as on the values of v_k in some extended spatial neighborhood, or on several previous cascade stages, and has been experimentally found to be a reasonable approximation by *Friedrich and Peinke* [1997]. That assumption also makes intuitive sense if v_{k+1} evolves faster, or on a smaller scale, than v_k, and is therefore in some kind of equilibrium with its precursor. If the cascade is deterministic in that sense, v_k can be represented as a product

$$v_k/v_0 = q_k q_{k-1} \cdots q_1. \tag{17}$$

in which the factors $q_k = v_k/v_{k-1}$ are statistically independent of each other.

If the underlying process is local, and invariant to scaling transformations, the transition probability density function has to have the form

$$p_T(v_{k+1}|v_k) = v_k^{-1} w(q_{k+1};\, k). \tag{18}$$

The multiplicative model works most naturally for positive variables, and we will assume that to be the case in the following, but most results can be generalized to arbitrary distributions. We will also assume for simplicity that all the cascade steps are equivalent, so that the distribution $w(q)$ of the multiplicative factors is independent of k, and depends only on our choice for the scale ratio r_{k+1}/r_k.

Local deterministic self-similar cascades lead naturally to intermittent distributions, in the sense that the high-order flatness factors for v_k become arbitrarily large as k increases. It follows from (16)–(18) that the n-th order moment for p_k can be written as

$$S_k(n) = \int \xi^n p_k(\xi)\, \mathrm{d}\xi = S_0(n) S_w(n)^k, \tag{19}$$

where $S_w(n)$ is the n-th order moment of the multiplicative factor q, and n is any real number for which the integral exists. If we define flatness factors as in (5), we can rewrite (19) as

$$\sigma_k(n)/\sigma_0(n) = \sigma_w(n)^k. \tag{20}$$

It follows from Chebichev's inequality that

$$S(n) \geq S(n-2)S(2) \geq S(n-4)S(2)^2 \ldots, \tag{21}$$

from where

$$1 \leq \sigma(4) \leq \sigma(6) \ldots, \tag{22}$$

which is true for any distribution of positive numbers. Equality only holds for trivial distributions concentrated on a single value. Except in that case, the expression in (20) increases without bound with the number of cascade steps, and the flatness factors diverge.

It is tempting to substitute k in (19) by a continuous variable, in which case the p.d.f.s form a continuous semi-group generated by infinitesimal scaling steps. This leads to beautiful theoretical developments [*Novikov*, 1994], but it is not necessarily a good idea from the physical point of view. For example, while it might be reasonable to assume that the properties of an eddy of size r depend only on those of the eddy of size $2r$ from which it derives, the same argument is weaker when applied to eddies of almost equal sizes. We will restrict ourselves here to the discrete case.

Limiting distributions

The multiplicative process just described can be summarized as a family of distributions $p_k(v_k)$ such that the probability density for the product of two variables is

$$p(v_{k_1} v_{k_2}) = p_{k_1+k_2}(v_{k_1+k_2}), \tag{23}$$

and it is natural to ask whether there is a limiting distribution for large k. We know that in the case of sums, rather than products, such distributions tend to Gaussian under fairly general conditions, and the first attempt to analyze (23) was to reduce it to a sum by defining

$$z = k^{-1} \log(v_k/v_0). \tag{24}$$

The argument was that z would tend to a Gaussian distribution, and that the limiting distribution for v_k would be lognormal.

This was soon shown to be incorrect [*Novikov*, 1971]. The central part of the distribution approaches lognormality, but the tails do not, because the central limit theorem does not apply to them. The family of lognormal distributions is a fixed point of (23), but it is unstable, and it is only attained if the individual generating distributions are themselves lognormal [*Jiménez*, 2000]. This contrasts with the situation for sums of random variables, in which the Gaussian distribution is not only a fixed point, but also has a very large basin of attraction.

Multifractals

The problem with using transformation (24) to find the limiting distribution of a multiplicative process is not so much the technique of analyzing the statistics of products in terms of those of sums, but the inappropriate use of the central limit theorem. It can be bypassed by using instead the theory of large deviations of sums of random variables. The key result is obtained by expanding the characteristic function of p_k when $k \gg 1$, and states that

$$p_k(v_k) \approx \left(\frac{-\phi_0''}{2\pi k}\right)^{1/2} \mathrm{e}^{k[\phi(z)-z]}, \tag{25}$$

where z is defined as in (24) and ϕ, which plays the role of an entropy, is a smooth function of z [*Lanford*, 1973]. Primes stands for derivatives with respect to z. Let us define z_n as the point where

$$\phi_n' \equiv \phi'(z_n) = -n, \tag{26}$$

which corresponds to the location of the maximum of $\phi + nz$.

The entropy ϕ can be computed from the moments of the transition probability density. Using Laplace's method to expand the n-th moment of p_k, we obtain

$$S_k(n) = \int_{-\infty}^{\infty} k\mathrm{e}^{k(n+1)z} p_k(v_k)\, \mathrm{d}z \approx \left(\frac{\phi_0''}{\phi_n''}\right)^{1/2} \mathrm{e}^{k(\phi_n + nz_n)}, \tag{27}$$

from where, using (19),

$$\lambda_n \equiv \log S_w(n) = \phi(z_n) + n z_n. \qquad (28)$$

The essence of Laplace's approximation is that, for $k \gg 1$, most of the contribution to the integral in (27) comes from the neighborhood of z_n, so that it makes sense to consider each such neighborhood as a separate 'component' of the cascade.

The geometric interpretation of this classification into components as a multifractal was developed in the context of three-dimensional homogeneous turbulence. The multifractal formulation assumes very little about the nature of each cascade step, but it is natural in turbulence to interpret it as the process in which eddies decay to a smaller geometric scale. The argument works for any variable for which scale similarity can be invoked, but we have seen that most experiments are done for the magnitude of the velocity increments across a distance r. If we assume for simplicity that $r_k/r_{k+1} = \mathrm{e}$, so that $r_k/r_0 = \exp(-k)$, equations (24) and (25) can be written as

$$v_k/v_0 = (r_k/r_0)^{-z_n}, \qquad p_k(z_n) \sim (r_k/r_0)^{-\phi_n}. \quad (29)$$

The multifractal interpretation is that the 'component' indexed by n, associated with the structure function $S(n)$, whose velocity increments are 'singular' in terms of r with exponent z_n, lies on a fractal whose volume is proportional to its probability, and which therefore has a dimension $D(z_n) = 3 + \phi_n$.

Note that (28) implies that the scaling exponents in (15) can be expressed as

$$\zeta(n) = -\log S_w(n) = -\lambda_n. \qquad (30)$$

The scaling exponents, the multifractal spectrum $D(z_n)$, the transition probability distribution $w(q)$, and the limiting distribution $p_\infty(v)$, unequivocally determine each other. Note that this implies that the only real information in (25) is the factor k in the exponent, which is required to recover the exponent k of the moments in (20) after many cascade steps. The rest of the expression for the p.d.f. is essentially a different notation for the scaling exponents. Note also that different quantities have different scaling exponents. For example, it follows from (4) that, if the scaling exponents for the local dissipation are $\zeta_\varepsilon(n)$, the exponents for Δu would be $\zeta_{\Delta u}(n) = n/3 + \zeta_\varepsilon(n/3)$.

Strictly speaking, and assuming that the solutions to the Navier–Stokes equations remain analytic at all times, the velocity gradients stay bounded at any finite Reynolds numbers, and the self-similar behavior of the structure functions cannot be continued beyond the viscous limit. Any discussion of the asymptotic behavior at very high orders is therefore limited to a hypothetical infinite-Reynolds number limit, in which the velocity field becomes singular. In that limit, several properties of the singularity can be derived from the previous discussion. If we assume, for example, that the multiplicative factor q is bounded above by q_b, which is reasonable for many physical systems, (24) implies that $z_n \le \log q_b$. In fact, if the transition probability behaves near q_b as $w(q) \sim (q_b - q)^\beta$ the scaling exponents tend to

$$\lambda_n = n \log q_b - (\beta + 1) \log n + O(1), \qquad (31)$$

for $n \gg 1$. In the case in which $w(q)$ has a concentrated component at $q = q_b$, the $\log n$ is missing in (31). In all cases the singularity exponent of the set associated with $n \to \infty$ is $z_\infty = \log q_b$, because the very high moments are dominated by the largest possible multiplier. In the case of a concentrated distribution, the dimension of this set approaches a finite limit, but otherwise

$$D(n) \approx -(\beta + 1) \log n, \qquad (32)$$

which becomes infinitely negative. This should not be considered a flaw. The set of events which only happen at isolated points and at isolated instants has dimension $D = -1$ in three-dimensional space, and those which only happen at isolated instants, and only under certain circumstances, have still lower negative dimensions. Sets with very negative dimensions are, however, extremely sparse, and are difficult to characterize experimentally.

The breakdown coefficients

The multifractal spectrum of the velocity differences in three-dimensional Navier-Stokes turbulence has been measured for several flows in terms of the scaling exponents, and appears to be universal. For reviews see *Nelkin* [1994]; *Sreenivasan and Stolovitzky* [1995], and the book by *Frisch* [1995]. From the discussion in the previous section it should be possible to derive from it the transition probability $p_T(q)$ of the cascade multipliers, but that turns out to be difficult. The relation lacks specificity. Transition models that are very different give very similar results, and it is impossible to choose among them using the available data [*Nelkin and Stolovitzky*, 1996].

A better approach is to measure directly the probability distributions $w(q)$ of the transition multipliers. Early attempts by *Van Atta and Yeh* [1975] and by *Chhabra and Sreenivasan* [1992] concluded that they are approximately independent of r, and that the multipliers of different cascade steps are also mutually independent. However, later experiments have shown that this was only a first approximation.

Consider for example the one-dimensional coarse-grained surrogate dissipation,

$$\varepsilon_r = \frac{1}{r} \int_{x-r/2}^{x+r/2} (\partial_x u)^2 \,\mathrm{d}x, \qquad (33)$$

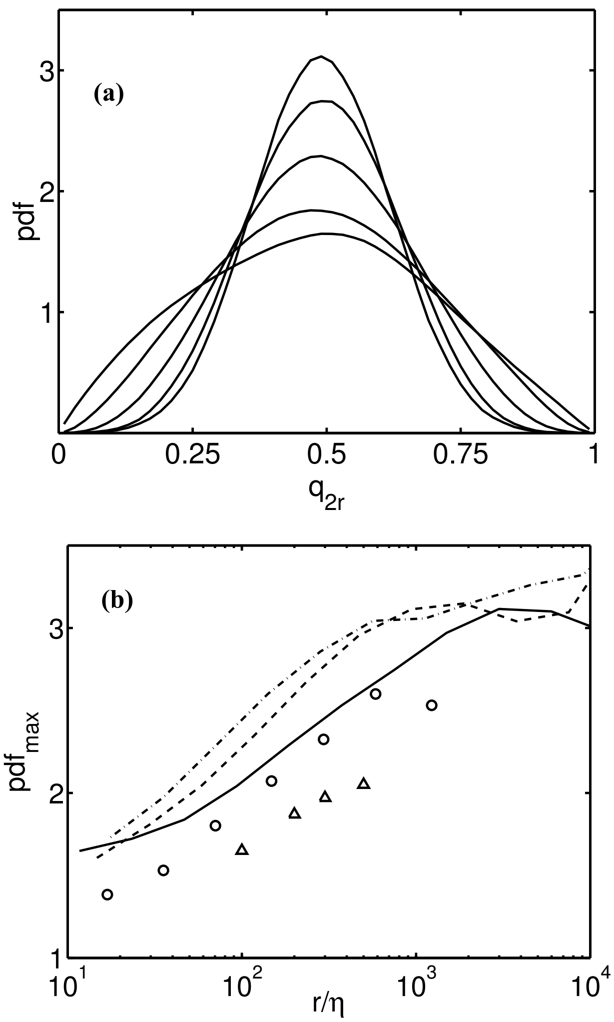

Figure 6. (a) The p.d.f.s of the breakdown coefficients of the surrogate averaged dissipation, for several averaging lengths. $Re_L = 1.7 \times 10^5$. In order of narrower distributions: $r/\eta = 10(\times 2)3000$. Data from *Belin et al.* [1997]. (b) Midpoint value of the p.d.f. as a function of averaging length. Various experiments, $Re_L = 1.6 \times 10^3 - 1.5 \times 10^7$. From *Jiménez et al.* [2000].

which was first used to define the breakdown coefficient by *Meneveau and Sreenivasan* [1991]. The p.d.f.s of the centered breakdown coefficients

$$q_{2r} = \frac{1}{2}\varepsilon_r/\varepsilon_{2r}, \tag{34}$$

are shown in Figure 6(a). They are bell-shaped in the inertial range, but they become wider at smaller separations. Figure 6(b), which includes the data from the previously mentioned early studies, shows that the maximum value of the p.d.f. varies continuously with the separation.

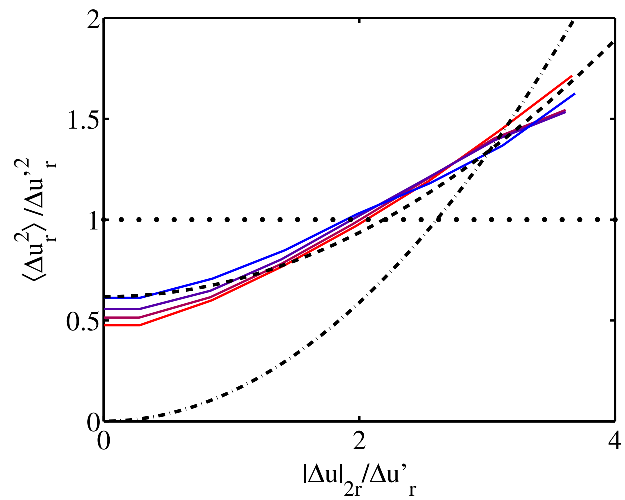

Figure 7. (a) Variance of the absolute value of the velocity difference across segments of size r, conditioned to the value of the velocity increment at size $2r$. The solid lines are $r/\eta = 140(\times 2)1100$, for $Re_L = 1.2 \times 10^6$. Data from the atmospheric surface layer by *Antonia and Pearson* [1999]. ----- , best fit to imperfect multiplicative model (35); —·— , best fit to pure multiplicative model; ········ , pure stochastic cascade.

Imperfect multiplicative processes

In fact, the Markovian hypothesis (16) does not necessarily imply a multiplicative cascade. The definition (18) of the transition probabilities requires both the scaling invariance of the equations, and locality in the sense that v_{k+1} depends only on v_k. It is possible to define Markovian invariant models that depend on more complicated functionals of v_k, such as on the statistical moments. The original *Kolmogorov* [1941] cascade, for example, is trivially Markovian, but the transition probability depends on v_{k+1}/v'_k, where v'_k is the global standard deviation of v_k. It was argued by *Jiménez and Wray* [1998]; *Jiménez* [2000], mostly on theoretical grounds, that such complications are natural when v is a field, rather than a scalar. The reason is that the cascade is most likely implemented by some instability that depends on the dynamics of the local structures, hence its dependence on v, but which is triggered or modified by the properties of the surrounding fluid, which is represented by v'.

Consider for example the velocity increments Δu_r. The simplest 'mixed' cascade model, incorporating both multiplicative and stochastic elements, is

$$\Delta u_r = \psi_1 \Delta u_{2r} + \psi_2 \Delta u'_{2r}, \tag{35}$$

where ψ_1 are ψ_2 are independent random processes. The limit $\psi_1 = 0$ is the purely stochastic cascade of *Kolmogorov* [1941], while $\psi_2 = 0$ is the multiplicative process discussed above. It follows from symmetry considerations that $\overline{\psi_2} = 0$ in homogeneous flows. The rest of the statistics have to be derived from experiments. One way is to examine the standard deviation of Δu_r, conditioned on a given Δu_{2r}. It follows from (35) that

$$\overline{\Delta u_r^2} - \overline{\Delta u_r}^2 \Big|_{\Delta u_{2r}} = \left(\overline{\psi_1^2} - \overline{\psi_1}^2\right) \Delta u_{2r}^2 + \overline{\psi_2^2}\, \Delta u'^2_{2r}, \tag{36}$$

and that

$$\Delta u'_r / \Delta u'_{2r} = \left(\overline{\psi_1^2} + \overline{\psi_2^2}\right)^{1/2}. \tag{37}$$

The conditional variance is therefore parabolic on the conditioning velocity difference. The apex of the parabola is on the horizontal axis for a purely multiplicative model, but it is off the axis for imperfect multiplicative models such as (35). The conditional variance becomes independent of Δu_{2r} for a purely stochastic cascade. Experimental results are given in figure 7 for a high-Reynolds number case in the atmospheric surface layer. They fit best a mixed model with

$$\overline{\psi_1} = 0.41, \quad \overline{\psi_1^2} = 0.25, \quad \overline{\psi_2^2} = 0.40. \tag{38}$$

The theory of processes such as (35) is not as well developed as those of either the purely stochastic or the multiplicative case. We summarize here some elementary results, but a fuller treatment has to be left for future publications.

The n-th order structure function of Δu_k is a polynomial of order k in the moments $\sigma_j(p) = \overline{\xi_j^p}$ of

$$\xi_j = \left(\overline{\psi_1^2} + \overline{\psi_2^2}\right)^{-1/2} \psi_j, \quad j = 1, 2. \tag{39}$$

The expansion for $S_k(n)$ is

$$S_k(n) \sim \sigma_1^k(n) + \text{lower order terms}, \tag{40}$$

where the expansion includes a constant term, independent of k. If all the moments $\sigma_1(p)$ are less than unity, the constant dominates for $k \gg 1$, and the statistics become regular, with finite moments. Otherwise, some of the moments diverge. We can use Chebichev's inequality to guarantee that, once a moment diverges, all the moments of higher orders diverge as well. For large k the structure functions then approach a power law, and the tails of the probability distributions behave as in multifractals. Note, however, that it may take longer to reach that limit than in a true multiplicative process, and that even approximate multifractality may not be reached in the limited cascades of the experimental Reynolds numbers.

From the experimental values in (38) it is impossible to say which would be the asymptotic limit of the cascade. The first two moments of the reduced stochastic coefficient are

$$\overline{\xi_1} = 0.51, \quad \overline{\xi_1^2} = 0.38. \tag{41}$$

What is required for some higher moment to exceed unity is that the p.d.f. of ξ_1 should have some non-zero mass above $\xi_1 = 1$. From the values in (41) that seems highly likely, and the experimental behavior of the structure functions strongly suggests that that is the case, but the result is weaker than in true multiplicative models.

Conclusions

We have reviewed the current understanding of the intermittent generation of very large velocity differences in turbulent fluids. In most cases the model of choice is the turbulent cascade, in which eddies become stronger as they decay to smaller sizes. The simplest model, a multiplicative process, has been well studied in the literature, and leads to multifractals. We have shown that other processes are possible, and that they fit the experiments better in some situations.

Discrete long-lived structures form near the dissipative range of scales, and they may dominate the statistics if they are singular enough. That is the case in decaying two-dimensional turbulence, in which the structures block the cascade. In the milder three-dimensional case, experiments favor a mixed multiplicative–stochastic cascade model, which would only tend to a multifractal in the limit of very high Reynolds numbers, probably beyond the reach of experimental, and even geophysical, flows.

Multiplicative or mixed cascades, and the resulting intermittency, are not limited to Navier-Stokes turbulence. The equations of motion have only entered our discussion through the assumption of scaling invariance. Multifractal models have in fact been proposed for many chaotic systems, from social sciences to economics, although the geometric interpretation is hard to justify in most of them. Many examples are given in *Schroeder* [1991].

It is also important to realize that the fact that a given process can in principle be described as a cascade does not necessarily mean that such a description is appropriate. Neither does a cascade imply a multiplicative process. For each particular case we need to provide a dynamical mechanism that implements both the cascade and the transition multipliers. In three-dimensional Navier-Stokes turbulence, the basic transport of energy to smaller scales and to higher gradients is vortex stretching. The differential strengthening and weakening of the vorticity under axial stretching and compression also provide a natural way of introducing the self-similar transition probabilities of the local dissipation.

Examples of non-intermittent cascades abound. Forced two-dimensional turbulence is dominated by an inverse energy cascade to larger scales, which is not intermittent. Conversely, we have already mentioned that the vorticity in decaying two-dimensional turbulence gets concentrated into stable vortex cores which eventually block the decay. The resulting enstrophy distribution is highly intermittent, but it is not well described by a cascade, nor by a multifractal.

In addition, the intermittency of some systems is not a small-scale effect in all directions. Turbulent mixing of a passive scalar, which is the key process in turbulent heat transfer and in the atmospheric dispersion of pollutants, is an extremely intermittent phenomenon. The gradients of the scalar tend to be very localized, but they concentrate in sheets, narrow in thickness but otherwise extended. Another problem in which intermittency is confined to large-scale surfaces is the motion of a three-dimensional pressureless gas, which has been used as a model for hypersonic turbulence and for the large-scale evolution of dark matter in the early universe.

In summary, the intermittent generation of extreme events is a fascinating property of many complex systems, including three-dimensional Navier-Stokes turbulence, which interferes, sometimes strongly, with their description by simple cascade models. It has different roots in different systems, but significant advances have been made in its quantitative kinematic analysis. In some cases we also have a qualitative understanding of the underlying physical processes. But in very few cases do we understand it well enough to make quantitative predictions.

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