DIRECT NUMERICAL SIMULATIONS OF ROTATING RAYLEIGH-BÉNARD CONVECTION

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1 Introduction

Convective heat transfer plays a key role in a wide range of physical phenomena and engineering applications. Rayleigh-Bénard convection is a classic showcase of convective heat transfer, mainly because of its accessibility to numerical and experimental analysis. In this type of flow, a layer of fluid is heated from below and cooled from above. The thermal expansion of the fluid then creates a buoyant force that leads to the convection of heat. The dynamics of Rayleigh-Bénard are characterized by two dimensionless parameters: the Rayleigh number \(Ra\) and the Prandtl number \(Pr\), defined as

\[
Ra = \frac{g\beta \Delta T L^3}{\nu \kappa},
\]

\[
Pr = \frac{\nu}{\kappa},
\]

where \(g\) is the gravitational acceleration, \(\beta\) the thermal expansion coefficient, \(\Delta T\) the temperature difference across the layer, \(L\) the height of the layer, \(\nu\) the kinematic viscosity and \(\kappa\) the thermal diffusivity. A critical Rayleigh number marks the onset of convection, beyond which the fluid layer is unable to maintain its hydrostatic equilibrium. Eventually, a turbulent flow develops for large Rayleigh numbers.

Additionally, in large-scale geophysical and astrophysical flows convective heat transfer is accompanied by forces generated by the rotation of the celestial body. Early research by Rossby (1969) has shown that the presence of rotation could increase the heat transfer with respect to the non-rotating case. This increase in heat transfer is associated with a qualitative change in the structure of the flow. The ratio of the inertial to the Coriolis force is described by the Rossby number, \(Ro\), defined as

\[
Ro = \frac{U}{2\Omega L},
\]

where \(U\) is the characteristic velocity and \(\Omega\) is the angular velocity. For high Rossby numbers the effect of rotation is limited and the flow is dominated by a large-scale circulation (LSC). For low Rossby numbers the Coriolis force becomes significant and is capable of breaking up the LSC. In that case, local vertical vortices occur that enhance the transport of heat.

Recent numerical and experimental studies by Kunnen et al. (2006, 2008, 2010) describe the influence of the Rossby number on the structure of the flow in more detail.

In this work we consider Rayleigh-Bénard convection in a rotating vertical cylinder with a width-to-height aspect ratio \(\Gamma = 1\). The goal is to study the effect of the Rayleigh and Rossby numbers on the structure of the flow and the heat transfer, with focus on the role of the Rayleigh number. We perform direct numerical simulations for fixed \(Pr = 6.4\), with \(Ra\) ranging from \(10^6\) to \(10^9\). For these simulations we deploy an open-source spectral-element code named nek5000, which has been developed by Fischer (1997, 2008). The code, written in MPI-parallel Fortran77/C, scales well to large numbers of parallel processes and received the 1999 Gordon Bell Prize for its algorithmic quality and scalability. The purpose of this work is essentially twofold. On the one hand, we study the performance of a spectral-element software package in this specific flow problem. On the other hand, we seek a deeper understanding of the physical phenomenon itself as it makes the transition from an unsteady laminar flow to a developed turbulent flow by changing the Rayleigh number.

2 Governing equations

This section describes stepwise the equations that govern Rayleigh-Bénard convection in a rotating cylinder.

Boussinesq approximation

Rayleigh-Bénard convection arises directly from the compressibility of the fluid: temperature differences between a hot and a cold plate cause a difference in the thermal expansion of the fluid near these walls according to the equation of state. These differences can set the fluid in motion as a result of gravitational forces. The governing equations can be simplified by the Boussinesq approximation, which in principle means a linearization of the equation of state. Basically, the effect of compressibility is neglected in every term except for the gravitational body force. A comprehensive derivation is given by Landau & Lifschitz (1987). Applying the Boussinesq approximation

\[
\]
to the conservation laws for Newtonian fluids yields the Boussinesq(-Oberbeck) equations,

\[ \frac{D\mathbf{u}}{Dt} = -\nabla p + \mu \nabla^2 \mathbf{u} + \rho \beta \nabla T \mathbf{g}, \]  
\[ \rho C_p \frac{DT}{Dt} = k \nabla^2 T, \]  
\[ \nabla \cdot \mathbf{u} = 0. \]

(4a), (4b), (4c)

Here, the reference density \( \rho \), the viscosity \( \mu \), the thermal expansion coefficient \( \beta \), the specific heat \( C_p \), and the thermal conductivity \( k \) are all assumed to be constant.

The Boussinesq equations are complemented by the boundary conditions that are listed in Table 1. The no-slip condition is imposed on the velocity field and the side-wall of the cylinder is assumed to be perfectly insulated.

Table 1: Boundary conditions

| Bottom plate | \( \mathbf{u} = 0 \) | \( T = T_0 + \Delta T \) |
| Top plate    | \( \mathbf{u} = 0 \) | \( T = T_0 \) |
| Side-wall    | \( \mathbf{u} = 0 \) | \( \frac{\partial T}{\partial n} = 0 \) |

Rotating coordinate systems

The effect of rotation is taken into account by adopting a co-rotating coordinate system and recasting Newton’s laws into this non-inertial coordinate system. This adaption to the new coordinate system introduces additional (fictitious) body forces. Quantities in the rotating system are denoted with a “prime”. Following Lanczos (1949), this change of coordinates gives the kinematic relations

\[ \mathbf{u} = \mathbf{u}' + \Omega \times \mathbf{r}', \]  
\[ \frac{d\mathbf{u}'}{dt} = \frac{d\mathbf{u}}{dt} + \Omega \times \mathbf{u}', \]

(5), (6)

where \( \mathbf{u}' \) and \( \mathbf{r}' \) are the velocity and position vector in the non-inertial coordinate system. Here, \( \frac{d\mathbf{u}}{dt} \) denotes the rate of change observed in the moving system. Differentiation of \( \mathbf{u} \) with respect to time gives

\[ \frac{d\mathbf{u}}{dt} = \frac{d\mathbf{u}'}{dt} + \Omega \times \mathbf{u} \]  
\[ = \frac{d\mathbf{u}'}{dt} + 2\Omega \times \mathbf{u}' + \Omega \times (\Omega \times \mathbf{r}'), \]

(7)

where the rotation rate \( \Omega \) is assumed to be constant in time. Substituting the total derivative into the Navier-Stokes equation (4a) yields

\[ \frac{D\mathbf{u}'}{Dt} = -\frac{1}{\rho} \nabla p' + \nu \nabla^2 \mathbf{u}' + \beta \nabla T' \mathbf{g}' \]  
\[ - 2\Omega \times \mathbf{u}' \]  
\[ - \Omega \times (\Omega \times \mathbf{r}'). \]

(8a), (8b), (8c)

Thus, the adoption of a non-inertial coordinate system leads to the appearance of two fictitious forces: the Coriolis force (8b) and the centrifugal force (8c).

Formally, one should first formulate the equations of motion in a rotating coordinate system, and then apply the Boussinesq approximation. This approach would introduce a density-dependent component of centrifugal force. Zhong et al. (2009) explained that this term can be neglected as the Froude number, defined as \( Fr = \Omega^2 L/(2g) \), is generally very small (\( Fr \ll 1 \)).

Dimensionless formulation

By convention we use the height of the cylinder \( L \) as the reference length and the free-fall velocity \( U = \sqrt{g\beta \Delta T L} \) as the reference velocity. The reference time scale is then \( L/U = \sqrt{L/(g\beta \Delta T)} \). With these primary reference parameters the Boussinesq equations take the non-dimensional form

\[ \left( \frac{\partial}{\partial t} + \tilde{\mathbf{u}} \cdot \nabla \right) \tilde{\mathbf{u}} = -\nabla \tilde{p} + \nabla \tilde{T} = \nabla^2 \tilde{\mathbf{u}} + \tilde{T} \]
\[ - \frac{1}{Ro} \mathbf{e}_z \times \tilde{\mathbf{u}} \]  
\[ - \frac{1}{4Ro^2} \mathbf{e}_z \times (\mathbf{e}_z \times \tilde{\mathbf{r}}), \]

(9a)

\[ \left( \frac{\partial}{\partial t} + \tilde{\mathbf{u}} \cdot \nabla \right) \tilde{T} = \frac{1}{PrRo} \nabla^2 \tilde{T}, \]

(9b)

\[ \nabla \cdot \tilde{\mathbf{u}} = 0, \]

(9c)

where the “tilde” denotes a dimensionless quantity. Solutions to the Boussinesq equations can then be characterized by the Rayleigh, Prandtl and Rossby numbers, respectively defined by (1), (2) and (3).

Also, the diffusivity coefficients are defined as

\[ \nu = \frac{\mu}{\rho}, \]

(10)

\[ \kappa = \frac{k}{\rho C_p}, \]

(11)

which are the viscous (or kinematic) and thermal diffusivity respectively. The dimensionless boundary conditions are listed in Table 2.

Table 2: Dimensionless boundary conditions

| Bottom plate | \( \tilde{\mathbf{u}} = 0 \) | \( \tilde{T} = 1 \) |
| Top plate    | \( \tilde{\mathbf{u}} = 0 \) | \( \tilde{T} = 0 \) |
| Side-wall    | \( \tilde{\mathbf{u}} = 0 \) | \( \frac{\partial \tilde{T}}{\partial n} = 0 \) |

Quantification of heat transfer

A main quantity of interest in Rayleigh-Bénard convection is the heat flux in the vertical direction. According to White (2006) conduction and convection contribute to the heat flux as follows

\[ q_z = \frac{k}{\kappa} u_z T - \frac{k}{\kappa} \frac{\partial T}{\partial z}. \]

(12)
Dividing by a factor $k \Delta T/L$ gives a dimensionless expression for the heat flux, also known as the Nusselt number,

$$Nu := \frac{q_x L}{k \Delta T} = \sqrt{PrRa} \tilde{u}_z \tilde{T} - \frac{\partial \tilde{T}}{\partial z}, \quad (13)$$

where the previous definitions of (1) and (2) are used. Using the boundary conditions from Table 2, we can measure the Nusselt number by averaging over the walls of the cylinder,

$$\langle Nu \rangle_w = \left\langle \frac{\partial \tilde{T}}{\partial z} \right\rangle_w. \quad (14)$$

Here, the operator $\langle \cdot \rangle_w$ denotes the average of a quantity over the walls of the cylinders. To obtain a better impression of the heat transfer in the bulk of the flow, the Nusselt number can also be evaluated by averaging over the total volume of the cylinder, denoted by $\langle \cdot \rangle_b$,

$$\langle Nu \rangle_b = 1 + \sqrt{PrRa} \left\langle \tilde{u}_z \tilde{T} \right\rangle_b. \quad (15)$$

Note that the Nusselt number indicates the ratio of convective to conductive heat transfer and that $Nu = 1$ in the case of pure conduction.

Finally, in the remaining sections we strictly adopt the dimensionless formulation and omit all the accents on the dimensionless quantities.

3 Spectral-element method

The Boussinesq equations (9) are solved by the spectral-element method (SEM) that is implemented in the code nek5000. The spectral-element method is essentially a variation of the finite-element method using higher-order piecewise polynomials as basis functions. Deville et al. (2004) describes in full detail the spectral-element method and its application to fluid dynamics.

The basis functions of the velocity are local tensor-product Lagrange interpolants of order $N$ on Gauss-Lobatto-Legendre nodes. On the other hand, the basis of the pressure are Lagrange interpolants of order $N-2$ on Gauss-Legendre nodes, which implies that the solution for the pressure is discontinuous at the element boundaries. Furthermore, an implicit-explicit third-order scheme is employed for time integration, in which the convective and additional force terms in the Navier-Stokes equation are treated by a third-order extrapolation scheme (EXT3) and the remaining terms by a backward differencing scheme (BDF3).

For the validation of the code we perform simulations for $Ra = 2 \cdot 10^8$, $Pr = 0.7$, $\Gamma = 0.5$ to compare with well-resolved simulations by Stevens et al. (2010), where we use a comparable spatial resolution. The spatial mesh consists of 42 240 elements with polynomial degree three. The total amount of grid points is then approximately 2.7 million, as each element accounts for $4^3$ degrees of freedom. A vertical resolution of 219 grid points is used. The difference in the Nusselt number remains below 5%, which confirms that the code is sufficiently accurate for predicting the heat transfer in Rayleigh-Bénard convection.

For the simulations of rotating Rayleigh-Bénard convection, we use a much higher spatial resolution. Here, the mesh consists of 519 616 elements of polynomial degree three, which amounts to approximately 33 million degrees of freedom. A vertical resolution of 391 grid points is used to resolve the horizontal boundary layers.

Symmetry-preserving discretization

In turbulent flows at high Rayleigh numbers the nonlinear convective operator gains considerable importance – its discretization proves to be a delicate matter. According to Verstappen & Veldman (2003), the discrete convective operator $C_{ij}$ should preserve the skew-symmetry, that is, the eigenvalues have to be purely imaginary. Loss of skew-symmetry inevitably introduces unwanted numerical dissipation and loss of energy. The SEM treats the convective term in the common variational formulation,

$$C_{ij} := \langle \phi_i, \mathbf{u} \cdot \nabla \phi_j \rangle,$$ \quad (16)

where $\phi$ denotes a basis function of the velocity. A danger hides, however, in the use of inexact quadrature rules for calculating the inner products. An ignorant choice of Gaussian quadrature for polynomials of order $N$ does not produce a perfectly skew-symmetric discretization. As illustrated by Rønquist (1996), the eigenvalues have minor real components, which initially might not have a practical impact on the accuracy of the numerical scheme. Nonetheless, this imperfection allows undamped error modes to persistently grow over time, which culminates in the blow-up of the numerical solution, eventually.

Malm et al. (2013) demonstrated that the skew-symmetry of the convective operator can be preserved by over-integration, which means applying quadrature rules with orders higher than $N$. Quadrature rules of order $3N/2$, instead of $N$, are generally sufficient to preserve the skew-symmetry of the convective operator up to machine precision. Over-integration is crucial for stabilizing the SEM in convection-dominated flows and is also adopted here.

4 Change in flow structure

In this section we illustrate how rotation affects the structure of the flow at various Rayleigh numbers, compared to the non-rotating case.

In Figure 1 and 2, snapshots of the temperature field and the vertical velocity field are shown. At $Ra = \infty$ (no rotation) the flow is dominated by a large-scale circulation at each Rayleigh number. The flow becomes gradually more turbulent if the Rayleigh number increases. The flow patterns at $Ro = 0.1$ are
qualitatively different. The large-scale circulation has disappeared and long vertical flow structures protrude from the bottom and top wall. This phenomenon is also known as *Ekman transport*, which moves the fluid away from the horizontal walls. Similar structures in the flow are observed by Kunnen *et al.* (2006, 2008, 2010). Horizontal cross-sections of the vertical velocity field are shown in Figure 3, which are taken from the middle of the cylinder \((z = 0.5)\). At \(Ra = 10^8\) the flow is too viscous to see a clear effect of rotation. At higher Rayleigh numbers more and more vortices appear and also reduce in size.

5 Increased heat transfer

In this section we quantify the heat transfer with the Nusselt number and show that rotation, in the particular case of \(Ro = 0.1\), increases the heat transfer by up to 19\% for a range of Rayleigh numbers.

The simulations are first run for 80 dimensionless time units \((4 \cdot 10^4\) time steps) in order to reach a statistically stationary state. The Nusselt numbers for \(Ro = \infty\) and \(Ro = 0.1\) are shown as function of time in Figure 4, in which the start-up stage is clipped. As expected, the Nusselt number increases with the Rayleigh number. Here, the influence of rotation leads to a slight increase in \(Nu\) for \(Ra = 10^8\) and \(Ra = 10^9\). Also, the figures show the volatile behaviour of \(\langle Nu \rangle_b\) indicating a strongly unsteady bulk flow, especially at higher Rayleigh numbers.

The time-averaged (wall) Nusselt number is plotted versus the Rayleigh number in Figure 5. The logarithmic plot reveals a power law between \(Nu\) and \(Ra\) for \(Ro = \infty\). Fitting a power law through the given data points yields \(Nu \sim Ra^{0.29}\), which agrees quite well to \(Nu \sim Ra^{2/7}\) from the scaling theory by Castaing *et al.* (1989). The scaling of \(Nu\) with \(Ra\) is described in more detail by the theory of Grossmann & Lohse (2000). For \(Ro = 0.1\) the scaling of \(Nu\) is not entirely given by a power law for all \(Ra\). However, asymptotically a comparable exponent seems to hold in a range beyond \(Ra = 10^8\). In this particular range the Nusselt number in the rotating case is increased by a maximum of 19\%. The increase in heat transfer is associated with the vertical vortices that are created by the rotation. On the other hand, this effect of rotation is less pronounced at lower Rayleigh numbers. For \(Ra = 10^6\), even a decrease of \(Nu\) is observed.

![Figure 1: Isocontours of temperature field at several Rayleigh and Rossby numbers. Red contour is \(T = 0.65\) and blue \(T = 0.35\).](image)
(a) $Ra = 10^6, Ro = \infty$
(b) $Ra = 10^6, Ro = 0.10$

(c) $Ra = 10^7, Ro = \infty$
(d) $Ra = 10^7, Ro = 0.10$

(e) $Ra = 10^8, Ro = \infty$
(f) $Ra = 10^8, Ro = 0.10$

(g) $Ra = 10^9, Ro = \infty$
(h) $Ra = 10^9, Ro = 0.10$

Figure 2: Isocontours of vertical velocity field at several Rayleigh and Rossby numbers. Red contour is $u_z = 0.07$ and blue $u_z = -0.07$.

(a) $Ra = 10^6, Ro = \infty$
(b) $Ra = 10^6, Ro = 0.10$

(c) $Ra = 10^7, Ro = \infty$
(d) $Ra = 10^7, Ro = 0.10$

(e) $Ra = 10^8, Ro = \infty$
(f) $Ra = 10^8, Ro = 0.10$

(g) $Ra = 10^9, Ro = \infty$
(h) $Ra = 10^9, Ro = 0.10$

Figure 3: Contours of vertical velocity field at several Rayleigh and Rossby numbers. Cross-section at $z = 0.5$. Red represents $u_z \approx 0.25$ at maximum and blue $u_z \approx -0.25$ at minimum.
6 Conclusions

The spectral-element method was found to be capable of accurate direct numerical simulations of rotating Rayleigh-Bénard convection.

The cases $Ro = \infty$ (no rotation) and $Ro = 0.1$ are studied for a range from $Ra = 10^6$ to $Ra = 10^9$. The rotation has a remarkable influence on the structure of flow. At $Ro = \infty$ the domain is filled with a single large-scale circulation, whereas at $Ro = 0.1$ vertical vortices are present, which increase in quantity and decrease in size with growing $Ra$.

For $Ro = \infty$, the scaling of $Nu$ with $Ra$ was found to be $Nu \sim Ra^{0.29}$, which agrees quite well to the theoretical $Nu \sim Ra^{2/7}$ from Castaing et al. (1989). For $Ro = 0.1$ a comparable scaling appears to hold in a range above $Ra = 10^8$. In this range, $Nu$ is increased up to 19% compared to the non-rotating case. The increased heat transfer is associated with the observed vertical vortices caused by the rotation.

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References


Lanczos C. (1949), The Variational Principles of Mechanics, University of Toronto Press.
