Global Analysis of Convective Instabilities in Nonparallel Flows

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Abstract
Convective instabilities in nonparallel flows are discussed. It is argued that the notion of a convective instability in the context of nonparallel flows holds only in an instantaneous sense. An approach for conducting global analysis of convective instabilities in nonparallel flows is presented. It is utilized to investigate several instabilities associated with bluff body wakes. Unlike the local analysis, the proposed method directly yields the global eigenmodes and does not involve approximations.

Keywords: Convective instability; Bluff body flows, Global linear stability analysis

1. Introduction
Landau and Lifshitz\textsuperscript{1} pointed out the existence of two types of instabilities in a convective medium: (a) absolute—the perturbation grows at the point of introduction and progressively contaminates the entire flow field and (b) convective—as the disturbance grows, it is carried away by the medium and ultimately leaves the flow field undisturbed. These notions were formalized and introduced in the case of plasma instabilities by Briggs\textsuperscript{2} and Bers\textsuperscript{3} and later applied by others to investigate hydrodynamic instabilities. Several researchers in the past studied the absolute/convective nature of instability of nonparallel flows via local analysis.\textsuperscript{4,5,6,7} In this approach, the flow profiles at different streamwise stations are analyzed by assuming that each profile corresponds to an independent parallel flow. The outcome is a categorization of the nonparallel flow field into regions which are absolutely or convectively unstable. It has been shown via parallel flow approximation, in bluff body flows, that a substantial region of the near wake becomes absolutely unstable before one observes a global response.\textsuperscript{5,8} This global response corresponds to the appearance of temporal instability. It can also be interpreted as an absolute instability of the nonparallel wake.

2. Convective instabilities, via local analysis, in nonparallel flows
Consider a station, in a nonparallel flow, which is found to be convectively unstable via the parallel flow approximation. A disturbance field nonzero over a sufficiently small region around the station, constructed from eigenmodes obtained via local analysis, is expected to convect as it grows with time. However, the original flow being nonparallel,
it encounters a different base flow as it convects to different streamwise locations. In fact, each streamwise location of this nonparallel flow is associated with different eigenmodes and growth rates, obtained via local analysis. This clearly shows that the predictions from the local analysis for a convectively instability in a nonparallel flow hold, at best, in an instanthenous sense. This is unlike the situation for an absolute instability, where the base flow seen by the instability is same at all times. Of course, the local analysis still assumes a parallel flow approximation at each station.

3. Global analysis in nonparallel flows

We present an approach for the investigation of absolute/convective nature of instabilities in nonparallel flows, without using a parallel flow approximation. In this approach, the entire flow field is considered, and therefore, the the global modes are obtained directly. As the global convective instabilities convect through a non-parallel flow they encounter a different base flow at each time instant. Therefore, although the approach is global in nature, one cannot dispense with the instantaneous nature of the conclusions obtained. The analysis, therefore, cannot be treated on the same footing as for asymptotic instabilities. We note that this situation is not an outcome of the shortcoming of the method. Rather, it is due to the nature of convective instabilities in a nonparallel flow.

3.1. Mathematical Formulation

Mittal and Kumar\(^9\) proposed a method for carrying out global analysis of convective instabilities in nonparallel flows. The idea follows from the fact that a convectively unstable flow appears absolutely unstable to an observer moving with the disturbance field. We therefore consider frames of reference moving with different speeds and in each of them carry out a temporal stability analysis to obtain the nature of absolute instability. To proceed further, two frames of reference are considered: a laboratory frame-\(x\) (=\((x, y, z)\)) which is at rest and another frame-\(z\) (=\((x', y', z')\)) that moves with velocity \(c\) (=\((c_x, c_y, c_z)\)) with respect to the frame-\(x\). The following sets of transformations relate the two frames:

\[
x = z + ct, \quad \nabla_x = \nabla_z, \quad \frac{\partial}{\partial t} \bigg|_x = \frac{\partial}{\partial t} \bigg|_z - c \cdot \nabla_z.
\]

Further, the perturbation field is considered to be a traveling wave of the form: \(u'(x, t) = \hat{u}(x - ct)e^{it}, \quad p'(x, t) = \hat{p}(x - ct)e^{it}\). It moves with velocity \(c\) in the laboratory frame while it appears stationary in the moving frame-\(z\). Substitution of this form of a perturbation, in the linearized equations for the disturbance fields for an incompressible flow lead to the following eigenvalue problem:

\[
\rho(\lambda \hat{u} + \hat{u} \cdot \nabla_z U + (U-c) \cdot \nabla_z \hat{u}) - \nabla_z \cdot \hat{\sigma} = 0, \quad \nabla_z \cdot \hat{u} = 0.
\]

Here \(\rho\) and \(U\) are the density and the base flow velocity, respectively. \(\hat{\sigma} = -\hat{p}I + \mu(\nabla \hat{u}) + (\nabla \hat{u})^T\), where \(\mu\) is the coefficient of dynamic viscosity. The boundary conditions used are the homogeneous version of the conditions on the base flow. Solution to this eigenvalue-vector problem can be found for different values of \(c\) and \(Re\). Here, \(\lambda = \lambda_r + i\lambda_i\) denotes the eigenvalue. \(\lambda_r\) is the growth rate of the disturbance while \(\lambda_i\) is the (circular) frequency. An unstable disturbance is associated with a positive value of \(\lambda_r\).

We note that a steady nonparallel flow \((U, P)\) in the laboratory frame appears unsteady \((U(z + ct)\) and \(P(z + ct)\) in the moving frame (frame-\(z\)). At \(t = 0\) the base flow in the moving frame is same as that in the laboratory frame. Hence, the same is used to carry out the stability analysis. In principle, the analysis can be carried out for a base flow at any time instant. However, as shown earlier, the conclusions for a convective instability for a nonparallel flow are valid in an instantaneous sense.

Another fact which deserves attention is that, since the global convective instability that we analyze appears before the onset of the temporal instability (global absolute instability), the growth of the convectively unstable disturbances must be due to the nonnormal nature of the stability operator obtained with \(c = 0\). For the same reason, the global convective instability could precede the occurrence of convective instability found via local analysis. The amplifications in the case of global convective instabilities are of course transient and need not be optimal. However, these fields travel at a certain speed and therefore are quite useful in relating to several phenomena which one encounters.
4. Results: Shear layer instability in flow past a circular cylinder

Several researchers in the past have investigated the instability of the separated shear layer for the flow past a circular cylinder,\textsuperscript{10,11,12,13} One finds that there is a large scatter in the critical value of $Re$ ($740 - 1900$) at which the instability is first observed. It was shown in a computational effort\textsuperscript{14} that the shear layer instabilities can be excited at values of $Re$ as low as 100. Mittal et. al.\textsuperscript{15}, using global linear stability analysis, showed that these instabilities are convective in nature and their existence could be traced down to $Re = 54$. At these low values of $Re$ they are quite weak and have relatively large scale structures. At higher $Re$ they develop small scale vortical structures which resemble the shear layer vortices observed in the experiments. Figure 1 shows a shear layer mode obtained via global linear stability analysis for the $Re = 500$ flow for $c_x = 0.25U$ and $c_y = 0$. To eliminate the primary wake instability, computation has been carried out for half a cylinder. The $Re = 500$ flow is associated with a large region of recirculating flow. The shear layer vortices arise along the separated shear layer as shown in Figure 1. Direct Time Integration of the flow equations confirm the growth rate as well as the time evolution of these vortices. Generally, convective instabilities are receptive to the environmental disturbances and their sensitivity increases with increase in $Re$. Depending on how careful one is in avoiding the environmental noise these instabilities could be observed at various values of $Re$. This observation is of great help in understanding why there is a large variation in the reported values of $Re$ at which the shear layer instability is first observed. The method has also been utilized to carry out the analysis for disturbance with non-zero $c_y$. It is found that the streamwise moving modes (with $c_y = 0$) are not necessarily the most dangerous modes.

References