Instability mechanisms in straight-diverging-straight channels

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Pressure-gradient-driven Poiseuille flow in a plane channel has been studied extensively as a prototype model for linear modal and non-modal instability mechanisms. A slight addition of an angle of divergence, $\alpha$, in the channel, modeled by the Jeffery-Hamel similarity solution of the equations of motion, drastically reduces the critical Reynolds number of the $\alpha = 0$ case. The first analysis of this kind was performed by Eagles\textsuperscript{1}, who used a local stability approach incorporating the parallel flow assumption. However, when the divergence angle increases, a global stability approach is essential in order to study the effect of channel geometry on the flow properties. Motivation for stability analyses in diverging channels with finite angles of divergence is derived from engineering applications where mixing is desired at very low Reynolds numbers. Knowledge of the stability characteristics in such micro-mixers can also enhance the potential for theoretically-founded flow control in such devices.

In order to explore the effect of angles of divergence on such flow systems the straight-diverging-straight (SDS) channel geometries were introduced for various values of the parameters $\alpha$ and expansion ratio ($\kappa = D/d$) i.e. the outlet height ($D$) to the inlet height ($d$). SDS flows were first studied experimentally by Sobey\textsuperscript{2} and theoretically by Saphira et al.\textsuperscript{3}, including the $\alpha = 90^\circ$ limit, denominated sudden expansion flow (SEF). Over a span of four decades, the sudden expansion flow has been studied theoretically and experimentally in much greater detail than those in SDS channels\textsuperscript{4,5,6,7,8,9,10}. A pitchfork type of a bifurcation was observed in SEF flows with ‘exchange of stability’, whereby the symmetric two-dimensional laminar flow breaks down with an increase in Reynolds number, $Re$, giving rise to an asymmetric steady laminar two-dimensional flow. With further increase in $Re$ the asymmetric flow becomes unsteady and oscillatory two- and three-dimensional flow ensues. The primary and secondary critical Reynolds numbers associated with two-dimensional flow instability have been documented in the above references. By contrast, the three-dimensional instability in the SEF geometry as well as two- and three-dimensional instability in the SDS geometries are far less studied. Shapira et al.\textsuperscript{3} were the first to employ modal linear global instability analysis to both SEF and SDS geometries and document the primary critical two-dimensional Reynolds numbers. However, very little is known about the secondary critical Reynolds number and the transition to turbulence in SDS flows. Swaminathan et al.\textsuperscript{11} carried out the global stability analysis in a given SDS geometry using a two-dimensional BiGlobal approach to determine the onset of two-dimensional unsteadiness. However, results were limited to small angle of divergences and nothing is yet known about modal three-dimensional perturbations or non-modal stability in both two- and three-dimensional SDS flows.

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The present work addresses SDS flows at finite angles of divergence and expansion ratios using both a modal and a non-modal linear instability analysis framework. Two open source codes, Nektar++ and Freefem++, based on a spectral/hp element and standard finite element method, respectively, have been used to obtain and cross-validate our results as regards both of base flow computation and stability analysis. The present results are independent of the grid size and a Richardson extrapolation has been carried out to demonstrate the convergence.

First, the SEF case has been cross-validated against the existing literature. Figure 1 shows the primary critical Reynolds number as a function of the expansion ratio, $\kappa$ at $\alpha = 90^\circ$. A reasonable agreement was found between our computations and several reference values in terms of the predictions of the primary critical Reynolds number. At low Reynolds numbers no laminar separation bubble exists in the basic two-dimensional steady state. With an increase in $Re$ the length of the separation bubble increases. With further increase of $Re$ the symmetric flow breaks down giving rise to a steady two-dimensional asymmetric flow. This trend was observed by Sobey\textsuperscript{2} experimentally and by Shapira et al.\textsuperscript{3} at a given set of parameters. Our results are in agreement with those obtained at the conditions examined by those references, thus confirming the symmetry breaking pitchfork type of bifurcation.

Information regarding the laminar separation zones provides an insight on the instability mechanisms expected in such channel flows. A systematic classification of the properties of the recirculation bubble as a function of $Re$, the angle of divergence, $\alpha$, and the expansion ratio, $\kappa$, has been performed. Figure 2 shows the percentage of maximum reverse flow $R = |U_{min}|/U_{max}|\%$, as a function of $Re$ and $\alpha$ for a given value of $\kappa = 2$. Results show that the flow may be prone to three-dimensional instability of the separation bubble region, via amplification of the stationary global mode of the two-dimensional laminar bubble\textsuperscript{12}, since the percentage of maximum reverse flow observed presently is as high as $R \approx 10 – 11\%$.

Our linear modal stability analysis confirms the prediction of a pitchfork bifurcation presented by Shapira et al.\textsuperscript{3}. The leading mode obtained in our analysis, shown in the Figure 3, is qualitatively similar for all the various angle of divergences. The analysis to determine the secondary critical Reynolds number based on the modal analysis, as well as a two-dimensional non-modal (transient growth) analysis is underway. Work has also commenced on the three-dimensional modal instability of SDS flows and will also be presented at the time of the conference.

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Fig. 2. (a) The maximum reverse flow, $R = |U_{\min}/U_{\max}|\%$ as a function of $Re$ for different $\alpha$ and expansion ratio, $\kappa = 2$ or $D/d = 2 : 1$. (b) The maximum reverse flow, $R = |U_{\min}/U_{\max}|\%$ as a function of $\alpha$ for different $Re$ and expansion ratio, $\kappa = 2$ or $D/d = 2 : 1$.

Fig. 3. The streamwise velocity component ($\hat{u}$) wall-normal velocity component ($\hat{v}$) and the pressure ($\hat{p}$) of the perturbations at angle of divergence, $\alpha = 45^\circ$, $Re = 120$ and expansion ratio, $\kappa = 2$ or $D/d = 2 : 1$.

References