A simple model that describes the non-linear dynamics of bladed-disk due to the dry friction exerted in attachment and the unsteady aerodynamic forced induced by the vibration of aerodynamically unstable airfoils is presented and analyzed. The analysis is focused on a simplified case whose dynamics is representative of a high aspect ratio low-pressure turbine rotor-blade. A parametric study is conducted to understand the dynamics of the the problem for times much longer than the fundamental period of the rotor blades, when the effect of the initial conditions is not relevant and a periodic steady state has been reached. It is concluded that the conjecture that the dynamics of multi-mode unstable bladed-disk, in the absence of external periodic forcing, finally reaches a state in which only a single traveling-wave exists is true and that this behavior may be reproduced using simplified models.

INTRODUCTION

Aeroengine low pressure turbines (LPTs) are made up of very slender and thin airfoils due to the steady trend to design very efficient, low cost, low weight turbomachinery. Cost and weight reductions are obtained by reducing the part count, increasing the lift per airfoil and designing light, high aspect ratio airfoils. The latter lowers the natural frequencies of the assembly, and therefore the reduced frequency, $k$, up to a point in which the airfoils may become aerodynamically unstable giving rise to the onset of flutter. Nowadays flutter may become a dominant constraint on the design of modern LPTs precluding the use of more efficient aerodynamic configurations.

The standard procedure to alleviate this problem is the use of inter-locked rotor blades where z-shaped shrouds are allowed to get in contact with the shrouds of the neighbouring airfoils, increasing the frequency and changing the mode-shapes of the assembly. An alternative approach is to weld the airfoils in pairs to obtain mode-shapes which provide a larger aerodynamic damping in a cantilever configuration. For a more in depth discussion of both configurations see [1].
When the rotor airfoils are aerodynamically unstable, the blade vibration grows up to a point in which the motion is non-linearly saturated by mechanical damping. Bladed-disks are usually assembled using contact attachments and the dissipation is mainly due to the dry friction that takes place in fir-tree or dove-tail attachments, although in principle other devices such as under-platform dampers or cover plates may contribute to increase the damping of the rotor blades. In any case what is important from and engineering point of view it is to estimate the vibration level of the rotor blade, that ultimately will determine the life of the component.

Figure 1 sketches the transient of an unstable rotor blade. The vibration amplitude of aerodynamically stable rotor blades under the absence of other perturbations is zero, however the amplitude of slightly unstable rotors grows exponentially during an initial linear stage and eventually becomes saturated by friction. The final vibration amplitude, which is obtained for times much larger than the inverse of the natural frequency of the unstable mode, depends on the ratio of the work-per-cycle exerted by the aerodynamics and the work-per-cycle dissipated by friction. When this ratio is small the bladed-disk may sustain indefinitely non-synchronous vibrations whilst if the ratio is high the rotor blade may fail very quickly. The former case may be seen in LPTs [2] while the latter is more typical of fans and compressor blisks [3, 4, 5].

The effect of friction on aerodynamically unstable rotor blades from a conceptual point of view was first studied by Sinha and Griffin [6, 7]. To clarify some of the basic issues addressed in the present investigation it is interesting to review the main results obtained from a single degree of freedom problem. If the unsteady aerodynamics is linear, then the unsteady pressure scales with the vibration amplitude, \( \delta \), and hence the aerodynamic work per cycle scales as the square of the amplitude, \( W_{\text{aero}} \propto \delta^2 \). The scaling of dry friction dissipation depends on the vibration amplitude. For very small vibration amplitudes (\( \delta < \delta_{\text{off-set}} \)) the rotor blade is stuck on the attachment and the dissipation is null, provided that the material structural dissipation is neglected. For large vibration amplitudes there is a macro displacement of the rotor blade in the attachment. The tangential force is constant and therefore the work dissipated per cycle is proportional to the displacement, \( W_{\text{macro}} \propto \delta \). Between both situations there is a regime, known as micro-slip, where only a fraction of the contact surface is sliding. Different models exist to describe this behaviour (see for instance the Midlin’s model [8] to describe the contact between two elastic spheres). What is important to highlight at this stage is that the work dissipated per cycle is of the form \( W_{\text{aero}} \propto \delta^n \), with \( n > 2 \), typically \( n = 3 \).

The situation is sketched in Fig. 2. The balance between the aerodynamic self-excitation and the dry friction provides either one or three solutions, depending on the relative value between them. The trivial solution, \( \delta = 0 \), is unstable and any small perturbation from \( \delta = 0 \), moves the system towards the solution 2, which is a stable cyclic limit. Solution 2 is an attractor and any perturbation of the cyclic limit comes back to the solution 2 unless we reach the amplitude \( \delta_1 \), which is an absolute stability limit, since the solution 3 is unstable. Alternatively, if the aerodynamic self-excitation is too large, or the friction work too low, the only solution is the trivial one and the system is unstable.

---

Figure 2. WORK PER CYCLE OF THE AERODYNAMIC FORCE (DASHED LINE) AND THE DRY FRICTION (SOLID LINE) AS A FUNCTION OF THE AMPLITUDE

Figure 3. ALTERNATING STRAIN-GAUGE AMPLITUDE AS A FUNCTION OF THE FREQUENCY AND SHAFT SPEED
There are computational [1] and experimental evidences (see Fig. 3) that indicate that, in spite of the stabilising effect of welding the blades in pairs, some LPT welded-in-pair rotor blades are aerodynamically unstable, although their vibration amplitude is well within admissible levels. Under these circumstances we are interested in predicting the vibration amplitude of the rotor blade to estimate the alternating stress field. Figure 3 shows engine measurements in a strain-gauge located in the shank of a LPT rotor blade. Apart from the synchronous excitation it is clearly seen a non-synchronous excitation of a certain mode in the whole range of the shaft speed. This is an indication that this mode aerodynamically unstable.

This paper describes for the first time the dynamics of a bladed-disk with several unstable nodal diameters of the same family in the absence of external periodic forcing. This means that the only aerodynamic forces are due to the self-excited motion of the airfoils. Special attention is paid both to the initial transient and the friction saturated final state. We describe first two fully coupled fluid-structure numerical tools. The first is a time-accurate method in which both the fluid domain and the structural model are marched in time simultaneously. The structural model is non-linear since it includes a friction model. The second is a novel method that assumes a pre-computed quasi-periodic aerodynamics and where only the structural equations are marched actually in time. This second method is much faster than the first and allows parametric studies. Both approaches have been used to analyze a simplified 2D problem whose dynamics retains all the characteristics of a realistic bladed-disk. The airfoil displacements are numerically integrated for very long times, what gives the simulations a unique nature also. Finally we have drawn some conclusions. First it has been demonstrated that the simplified method is accurate enough to deal with this type of problem and that the unsteady aerodynamics is linear. Secondly we have shown that in the long term only the most unstable of the modes survives, the rest are damped by non-linear mechanisms.

ANALYSIS METHODOLOGY

The governing equations for the solid domain of an aeroelastic system may be expressed in compact form as

\[ [M] \ddot{x} + [C] \dot{x} + [K] x + \{G(x)\} = \{F_a(x,t)\} \]  

where \([M]\), \([C]\) and \([K]\) are respectively the mass, damping and stiffness matrices, \([G]\) the vector of non-linear and (in general) dissipative forces associated to the fir-tree dry friction and \(F_a\) the vector of aerodynamic forces. The aerodynamic forces are in general computed by solving the RANS equations, either considering the forcing associated to the periodic excitation of the incoming wakes or the unsteady pressure associated with the vibration of the airfoils, as it is the case in the present study.

Two different approaches have been considered to deal with this system of equations: a fully-coupled methodology, where the

**NOMENCLATURE**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a_{FT})</td>
<td>Half-width of the fir-tree</td>
</tr>
<tr>
<td>(A)</td>
<td>Cross area of the rotor blade</td>
</tr>
<tr>
<td>(b_{FT})</td>
<td>Half-length of the fir-tree</td>
</tr>
<tr>
<td>(c)</td>
<td>Axial chord</td>
</tr>
<tr>
<td>(E)</td>
<td>Elastic modulus</td>
</tr>
<tr>
<td>(F)</td>
<td>Force</td>
</tr>
<tr>
<td>(FE)</td>
<td>Finite Element</td>
</tr>
<tr>
<td>(G)</td>
<td>Shear modulus</td>
</tr>
<tr>
<td>(I)</td>
<td>Moment of inertia of the section</td>
</tr>
<tr>
<td>(IBPA)</td>
<td>Inter-blade Phase Angle</td>
</tr>
<tr>
<td>(k)</td>
<td>Reduced frequency</td>
</tr>
<tr>
<td>(K_1, K_2)</td>
<td>Linear and non-linear stiffness at the root</td>
</tr>
<tr>
<td>(L)</td>
<td>Beam or rotor blade length</td>
</tr>
<tr>
<td>(LPT)</td>
<td>Low-Pressure-Turbine</td>
</tr>
<tr>
<td>(M)</td>
<td>Mach number</td>
</tr>
<tr>
<td>(N)</td>
<td>Normal load</td>
</tr>
<tr>
<td>(Re)</td>
<td>Reynolds number</td>
</tr>
<tr>
<td>(s)</td>
<td>Spacing</td>
</tr>
<tr>
<td>(SG)</td>
<td>Strain-Gauge</td>
</tr>
<tr>
<td>(t)</td>
<td>Time</td>
</tr>
<tr>
<td>(U_{ref})</td>
<td>Characteristic velocity of the mean flow</td>
</tr>
<tr>
<td>(x)</td>
<td>Deflection of any degree of freedom or the rotor</td>
</tr>
<tr>
<td>(y)</td>
<td>Deflection of the rotor blade fir-tree</td>
</tr>
<tr>
<td>(z)</td>
<td>Coordinate along the beam or airfoil</td>
</tr>
<tr>
<td>(\delta_a)</td>
<td>Characteristic vibration amplitude of the airfoil</td>
</tr>
<tr>
<td>(\mu)</td>
<td>Friction coefficient</td>
</tr>
<tr>
<td>(\lambda)</td>
<td>Penetration length</td>
</tr>
<tr>
<td>(\Lambda)</td>
<td>Aspect ratio</td>
</tr>
<tr>
<td>(\rho)</td>
<td>Density of the mean flow</td>
</tr>
<tr>
<td>(\tau)</td>
<td>Non-dimensional time</td>
</tr>
<tr>
<td>(\rho_0)</td>
<td>Density of the rotor blade</td>
</tr>
<tr>
<td>(\rho_a)</td>
<td>Characteristic air density</td>
</tr>
<tr>
<td>(\omega_N)</td>
<td>Natural frequency</td>
</tr>
<tr>
<td>(a)</td>
<td>Aerodynamic or airfoil</td>
</tr>
<tr>
<td>(f)</td>
<td>Friction</td>
</tr>
<tr>
<td>(f - NL)</td>
<td>Non-linear component of friction</td>
</tr>
<tr>
<td>(L)</td>
<td>Loading branch</td>
</tr>
<tr>
<td>(U)</td>
<td>Unloading branch</td>
</tr>
</tbody>
</table>
fluid and solid domains of the aeroelastic system are solved simultaneously; and a semi-uncoupled methodology, where a simplified approach is adopted to simulate the effect of the aerodynamics.

**Fully-Coupled Method**

An unsteady non-linear unstructured RANS code, known as $\mu^2 s^2 T$ [9, 10], is used to perform the time-domain fully-coupled simulations, specifically to compute the aerodynamic forcing term $\{F_a(x,t)\}$. The main characteristics of the codes directly related to the unsteady aero-mechanical simulations are:

1. The time integration of the RANS equations is performed using a Dual Time Step algorithm that uses a multigrid technique as relaxation method for the implicit set of equations. This allows the use of temporal steps much larger than the dictated by the CFL limit of the explicit scheme.

2. The time integration of the structural set of equations is performed using a 3rd order, 4-stages, SDIRK Singly-Diagonally-Implicit Runge-Kutta (SDIRK) algorithm. The main advantage of this method when compared to other popular algorithms, such as the second order Newmark-$\beta$, is that it is L-stable, i.e., automatically damps very high frequency modes, instead of artificially conserving the energy for such modes. On the other hand the algorithm is not conservative, with fourth order dissipation, so in theory care has to be taken in order to keep the numerical dissipation negligible when compared to the physical damping. Nevertheless that is not a serious issue in practice and does not impose a more severe time-step restriction than phase error considerations. The resulting set of non-linear equations are solved using a Newton-Raphson method.

3. A moving grid approach has been adopted in the RANS solver to accommodate the displacements at the mechanical surface derived from the structural solver. A Laplacian smoothing is used to transfer these displacements to the inner nodes of the fluid grid; a multi-grid method is used as well to accelerate the convergence of the grid smoothing.

Every time step the RANS, structural and grid displacement equations are solved simultaneously. Due to the iterative nature of all the individual solvers their synchronization is simple.

**Semi-Uncoupled Method**

With the aim of drastically reducing the computational cost with a minor modification of the results we have developed the so-called Semi-Uncoupled Method (SUM). In this approach the aerodynamic forces, which are the most expensive part of the analysis, are precomputed using the linearized version of the $\mu^2 s^2 T$, code known as $\mu^2 s^2 T - L$, [11] in the frequency domain and in a traveling-wave basis. However these forces could be precomputed as well using a non-linear aerodynamic computation although anyhow the hypothesis of linearity is needed later on. The two main underlying hypotheses are:

1. The unsteady aerodynamics associated to the vibration of the airfoils is linear. This true if the vibration amplitude is much smaller than the chord, $\delta/c \ll 1$, and there are not shock waves in the base flow.

2. The airfoil motion is harmonic with a constant vibration amplitude.

The hypothesis of linearity is both easy to under consider the results from a linearized analysis for the aerodynamic forces; this would be appropriate as long as the perturbations around a aerodynamic steady state are small. By using the linear CFD solver $\mu^2 s^2 T - L$ we can obtain an expression for the aerodynamic forces in the frequency domain

$$\{F_a(x,t)\} = \Re(\{\hat{P}_a\} e^{i\omega t} + |D(\omega)|\{x\} e^{i\omega t}) \quad (2)$$

Here, $\{\hat{P}_a\}$ is a complex vector and corresponds to the forcing due to unsteady external forces (i.e., not directly due to the vibration, such as stator-rotor interactions).

$|D(\omega)|$ is a complex matrix and has the role of an influence coefficient matrix, relating the DOF motion and the unsteady forces. Notice the dependence on the vibration frequency $\omega$. In order to circumvent that dependence and include the related information into the time-domain equations we consider a reference frequency $\omega_0$, and then we define

$$[K_a] = \Re(|D(\omega_0)|)$$

$$[C_a] = \frac{1}{\omega_0} \Im(|D(\omega_0)|)$$

Using the previous definitions we construct the following system of equations

$$[M]\{\ddot{x}\} + ([C] + [C_a])\{\dot{x}\} + ([K] + [K_a])\{x\} + \{G(x)\} = \{P_d(t)\} \quad (3)$$

If we pre-compute the aerodynamic external forces $P_d(t)$ we can integrate the aeroelastic system with no need to include feedback.
with a CFD code. The time integration of the resulting system is performed using the same SDIRK algorithm that in the fully-coupled methodology.

Notice that this approach implies the assumption that the DOF motion is purely harmonic with a frequency \( \omega_0 \) known in advance.

**POST-PROCESSING STRATEGY**

The output from any of the methodologies is the evolution of the different DOFs (both displacement and velocity) in the time domain. Nevertheless, typically most aeroelastic studies are performed in the frequency domain. In order to easily compare both approaches some spectral analysis of the time signals is required.

The most straightforward technique involves the use of Hamming windowing and Fast Fourier Transforms (FFTs) in the same that are use in physical experiments. Nevertheless, the accuracy in the prediction of the frequency (and amplitude), even in purely harmonic signals, is limited by the width of the temporal window.

An alternative approach, useful for signals with a clear tonal component, is the fitting of functions of the form \( \sum A_i \cos(\omega t + \Phi_i) \), i.e.: a collocation method. Such a fitting is feasible as long as the noise level is low, and relies on using as a first guess the results from the direct FT. This technique provides very accurate results for both frequency and amplitude for each of the different tones.

The dynamic of structures with cyclic symmetry may only be fully appreciated if the solution is expressed in either traveling-wave or standing-wave form. For models considering sector based DOFs (i.e., not considering directly a set of cyclic modes as the basic DOF in the formulation) additional post-processing is required. The expansion of the solution in standing waves (nodal diameters) is straightforward and only requires a spatial DFT for the homologous DOFs at different sectors. The extraction of the traveling waves is based on the hypothesis that the motion is harmonic at the angular frequency \( \omega = \omega_0 \). If we consider the forward and backward traveling waves with the \( N^{th} \) nodal diameter we have:

\[
\begin{align*}
x(t) &= A \cos(\omega_0 t - N\theta) + B \sin(\omega_0 t - N\theta) \\
V(t) &= -\omega_0 A \sin(\omega_0 t - N\theta) + \omega_0 B \cos(\omega_0 t - N\theta)
\end{align*}
\]

By considering \( x_c \) and \( V_c \), the cosine components of the displacement and the velocity respectively of the DOFs at a given instant, and \( x_s \) and \( V_s \), the sine components for the traveling-wave amplitudes at a given instant:

\[
\begin{align*}
A &= \frac{1}{2} \sqrt{(x_c - \frac{1}{\omega_0} V_c)^2 + (x_s + \frac{1}{\omega_0} V_c)^2} \\
B &= \frac{1}{2} \sqrt{(x_c + \frac{1}{\omega_0} V_c)^2 + (x_s - \frac{1}{\omega_0} V_c)^2}
\end{align*}
\]

**MODEL TEST CASE**

The main purpose of the present work is to increase the understanding of the dynamics of bladed-disks in presence of several unstable modes. For this reason a fairly simple mechanical model has been used to represent the bladed-disk. The model represents a wheel of rotor blades attached by means of a fir-tree to an ideally infinitely stiff disc. The model considers just two DOFs per sector. The first DOF of the \( j^{th} \) sector, \( x_j \), corresponds to a bending mode of the airfoil whose motion follows a direction orthogonal to the engine axis, as it is sketched in Fig. 4. The second DOF, \( y_j \), is associated with the micro-displacements of the fir-tree.

Since the disk stiffness is considered infinite, no mechanical coupling between different sectors is included. The 'fir-tree DOF' is attached to the solid disk considering a non-linear micro-slip friction model. In the limit where no displacements appear at
the fir-tree, the natural frequency of the blade-alone analysis, \( \omega_a^2 = K_a/M_a \), is recovered.

The governing equations for \( j \)-th sector are (see Fig. 4):

\[
M_j \ddot{x}_j + K_a(x_j - y_j) = F_a(\ldots, x_{j-1}, x_j, x_{j+1}, \ldots) \\
M_j \ddot{y}_j + K_a(y_j - x_j) + K_f y_j = F_{f-NL}(y_j)
\]

(6)

where \( F_a \) is the aerodynamic force acting on the \( j \)-th airfoil, that depends on displacements of all the airfoils, and \( F_f = -K_f y_j + F_{f-NL} \), is the friction force in the fir-tree decomposed in its linear, \( -K_f y_j \), and non-linear, \( F_{f-NL} \), components, which uses the non-linear friction model by Olofsson [12] whose expression for the loading branch of the hysteresis loop is:

\[
F_f = F^L + 2\mu t N \left( 1 - \left( \frac{2G' y_j - y^L}{\mu t E'_l} \right)^{\frac{1}{2}} \right),
\]

(7)

Here \( F^L \) is the force at the end of the previous unload branch and \( y^L \) is the displacement at the end of the previous unload branch.

On the other hand, for the unloading branch we have:

\[
F_f = F^L - 2\mu t N \left( 1 - \left( \frac{2G' y_j - y^L}{\mu t E'_l} \right)^{\frac{1}{2}} \right)
\]

(8)

where \( F^L \) and \( y^L \) are the values at the end of the previous load branch.

The system of equations 9 may be written in non-dimensional form as:

\[
\ddot{x}'_j + (\ddot{x}_j - \ddot{y}_j) = \ddot{x}' \zeta_d(\ldots, x_{j-1}/x_j, x_{j+1}/x_j, \ldots, x_j, k) \\
\theta \ddot{y}'_j + \theta \ddot{x}_j + (\ddot{y}_j - \ddot{x}_j) = f \cdot \bar{F}_{f-NL}(\bar{y}_j, \tau)
\]

(9)

where the primes denote derivation with respect the non-dimensional time, \( \tau = t\omega_a \), \( \ddot{x}' \zeta_d = \ddot{x}' = M_a/\omega_a^2 \), is the non-dimensional aerodynamic force, \( \omega_f^2 = K_f/M_f \) the angular frequency of the non-linear spring considering the airfoil clamped, \( \theta = \omega_a^2/\omega_f^2 \), the ratio of the airfoil and fir-tree natural frequencies, \( f = \mu N/M_f \omega_f^2 \), the scaling of the non-linear component of the fir-tree spring and \( \bar{F}_{f-NL}(\bar{y}_j, \tau) \) the non-dimensional expression of the friction model, where the explicit dependence with \( \tau \) has been included to indicate the existence of history effects.

There are several key aspects of the model that deserve special attention:

1. The only non-linearity of the model is associated to the attachment between the rotor blade and the disk since the aerodynamic is linear although it may be eventually computed using a non-linear method.
2. The only coupling between different blades occurs through the aerodynamic interaction.
3. The only coupling between traveling-waves occurs through the non-linearity of the ‘fir-tree DoF’

It is important to recall these three points to interpret the results of the model.

The parameters of the model have selected to be representative of the dynamics of bladed-disk of a low-pressure turbine with the rotor blades welded-in-pairs. This fact affects only the aerodynamic parameters of the model. This effectively means that we are computing sectors that contain two airfoils, although only one out of two airfoils of the pair is displayed in Fig. 4. When we precompute the aerodynamics in the Semi-Uncoupled method, we use as well the aerodynamics of welded-pair rotor. The pairs of airfoils are assumed to move as a rigid body which is a realistic assumption [1].

**RESULTS**

Based on the previous model a set of different cases has been run using both methodologies. A free flutter numerical experiments is considered, i.e.: there is no external periodic forcing applied in the simulation, but there are unstable normal modes in the simulation. The main parameters of the cases (aerodynamics, mass, stiffness, friction constants) have been set to reproduce a real rotor; in particular, the exit Mach number is 0.57, the reduced frequency \( k \) is 0.12 and the critical damping ratio \( \xi \) for the most unstable mode is roughly 1.6\%. The unsteady aerodynamic simulations consider the 2-D mid-section of the rotor; the grid for each passage contains approximately 7000 points. Taking into account the comparatively low frequencies of the vibration, the turbulence is simulated applying the algebraic Baldwin-Lomax model with a quasi-steady approach.

Linear analyses for these cases get the damping results which can be seen in Fig. 5.

The unsteady aerodynamics resulting from the selected set of parameters may be seen in Fig. 5. It may be seen that the system is highly unstable due to the selected reduced frequency \( k = 0.12 \) which is very low and that several modes are unstable, the actual
number depending on the number of airfoils of the configuration. A slight discontinuity may be seen around IBPA = 0° due to the acoustic resonances.

**Six-Sector Computational Domain**

The first test case considers a domain with six sectors (i.e. 12 passages). The objective of simulating just six sectors is to observe the dynamics of a system with just two unstable modes. In this case we may see two pairs of backward and forward TWs for the ND = 1 and 2 plus the TWs corresponding with the ND zero and three, which are indistinguishable from the corresponding standing waves, totalizing six TWs. It may be appreciated in Fig. 5 that the forward TWs corresponding with the ND = 1 and 2 are unstable but with a different strength ($\xi_{ND=1}/\xi_{ND=2} \simeq 2.7$) while the rest are stable. This different growth rate of the two modes eases the derivation of conclusions and their tracking during the temporal evolution. For the initial condition of the simulation a single pair of blades is slightly displaced from its equilibrium position; this generates a perturbation in all the different TWs.

The vibration amplitudes of the airfoil DoFs using both methodologies are shown in Fig. 6. The time scale is normalized with the nominal vibration period of the rotor blades (i.e.: the period obtained linearizing the fir-tree constraint and neglecting aerodynamic terms), making the horizontal axis roughly equivalent to the number of vibration periods. The vibration amplitude is normalized with an analytical estimate of the maximum vibration amplitude performed using the methodology described in [2]. In practical terms the reader may think that the vibration amplitude has been normalized using the maximum theoretical vibration amplitude. Notice that both simulations get remarkably similar results; after an initial development stage with an exponential growth of the amplitude, the vibration saturates at a constant amplitude. The difference in the saturated amplitude prediction between both methods is roughly 7%, and in the saturation time slightly lower than 15%.

Figure 7 shows the solution post-processed using the wave-splitting described in Eq. 5; this is a good approximation with the evolution of the traveling waves. Lines with different colours denote the evolution of different TWs. The solid lines show the evolution of the TWs using the fully coupled approach while dashed lines denote the simulation perform using the SUM approach. It may be noticed the nearly exponential growth/decay during the early stages (5-7 first periods) of the simulation; this is in good agreement with classic linear aeroelasticity theory. In table 1 The critical damping ratio for the one nodal diameter forward traveling wave, obtained from the growth rate in the first periods of the simulation, can be compared with the results from a classical linear methodology. The agreement is nearly perfect for the semi-uncoupled methodology, but the differences with the fully coupled method are roughly 15%. The most unstable mode corresponds to the ND=+1. This is consistent with the results displayed in Fig. 7.

---

**Figure 5. CRITICAL DAMPING RATIO INCLUDED IN THE AEROELASTIC MODEL AS A FUNCTION OF THE IBPA OR THE NODAL DIAMETER**

**Figure 6. TIME EVOLUTION OF THE BLADE DoFS IN A SIX-SECTOR COMPUTATIONAL DOMAIN**
Figure 7. **TIME EVOLUTION OF THE TRAVELING-WAVE AMPLITUDES IN A SIX-SECTOR COMPUTATIONAL DOMAIN**

Table 1. **SIX-SECTOR COMPUTATIONAL DOMAIN: CRITICAL DAMPING RATIO FOR THE MOST UNSTABLE TRAVELING WAVE**

<table>
<thead>
<tr>
<th>Approach</th>
<th>$\xi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fully Coupled</td>
<td>1.79%</td>
</tr>
<tr>
<td>Semi-Uncoupled</td>
<td>1.53%</td>
</tr>
<tr>
<td>Linear</td>
<td>1.56%</td>
</tr>
</tbody>
</table>

The behaviour of the traveling wave with nodal diameter zero (black line in Fig. 7) requires further explanation. It must be taken into account that a steady aerodynamic force (lift) with a spatial pattern corresponding to the $ND = 0$ appears in the simulation. This force was not taken into account in the determination of the initial location of the airfoils and in the long run it induces an offset between the initial and final position of the airfoils, rather than an oscillation in the zero nodal diameter, which justifies the weird behaviour shown in the graph.

As the vibration amplitudes increase non-linear effects begin to play a greater role (10-30 periods into the simulation). In general, non-linear terms allow the interaction of patterns with different nodal diameters. So, even though most waves should decay according to the linear theory, once the most unstable mode reaches a large enough vibration amplitude the system does not behave any longer linearly. Notice however that, in any case, the vibration amplitudes of the stable waves are 2-3 orders of magnitude smaller than the amplitude of the most unstable mode.

Close to the saturation (30-60 periods into the simulation) the growth rates for the unstable modes decrease. Interestingly, while the most unstable mode tends towards a constant amplitude, the energy of the other unstable mode is drained and its amplitude limited to a fairly small value. This process is still active even after the envelope of the physical amplitude of the blades has already reached saturation (about period 40). This means that there is a non-linear process that adjusts the vibration phase of the airfoils at a fairly constant vibration amplitude.

This conclusion is quite remarkable since although in principle it could be thought that the saturated solution could be made up of a blend of the two unstable modes, the non linearity of the fir-tree filters out the second and transfer its energy to the most unstable mode. The conjecture is that when more than two unstable modes exist a similar mechanisms will concentrate all the energy in a single mode as well.

Once the amplitude is saturated and most of the energy (more than 99.9%) is concentrated into the most unstable mode the motion is nearly periodic. The results of performing a FFT using a sampling window of roughly 6 periods can be seen in figure 8. Notice that for the blade DoF the motion is nearly sinusoidal, with negligible presence of higher harmonics. On the other hand, the fir-tree DoF which is directly connected to the non-linear constraint shows significant values of the first odd harmonics. Using regression the best fitting for the blade frequency has been found to be 2.61% higher than the nominal vibration frequency.
36-Sector Computational Domain

This case is identical to the six-sector model in the sense that both, the aerodynamic and structural parameters are the same, the sole difference is that we are including in the model thirty-six sectors that physically corresponds to 72 rotor blades welded in pairs instead of six. Since we have now more DoFs the dynamics of the problem is richer and more complex. The unsteady aerodynamics associated to this case is also that displayed in Fig. 5, however since in this configuration we may accommodate more modes we have consequently more unstable modes. Actually the forward traveling waves corresponding to the \( ND = 2 - 13 \), both included, are unstable. These twelve unstable modes have different critical damping ratio as it may be appreciated in Fig. 5.

Figure 9 shows that the vibration amplitude of the airfoils saturates at nearly the same value than the model with six sectors. The same is true for the time needed to reach the amplitude saturation value. This effectively means that the growth rate of the instability in both cases is roughly the same, as could be expected, since the six-sector model contains nearly the most unstable mode.

Nevertheless, a more in depth analysis based on the tracking of the traveling-wave amplitudes (see Fig. 10) reveals that, even if the airfoil amplitudes have reached their final saturated value, there is still energy exchange among the different unstable waves. This is clearly seen between the non-dimensional time 100 and 150, where it may be appreciated that many modes are still active. Every line of Figs. 10 and 11 denotes the evolution of an aeroelastic mode (i.e.: a TW with a prescribed wave-length or ND). The colour code is consistent between both figures, unfortunately since every figure has 36 lines it is not appropriate to include the legends in them. The reader may pay attention to the colour code to compare the different figures.

The solution provided by both, the fully coupled and the semi-uncoupled approaches are very similar, not only in terms of the vibration amplitude of the airfoils but also in terms of the mode content. The comparison provided by Fig. 10 between both methods shows that not only qualitatively, but also quantitatively, the evolution of the corresponding modes, which are marked with the same colour, is very similar. This is a very good indication that the semi-uncoupled approach is working properly.

Figure 5 compares the aerodynamic damping of the system obtained by three independent methods: (i) The \( Ma^2s^2T - L \) frequency domain linear solver prescribing the frequency that is obtained assuming that the non-linear contact is fixed (solid line), (ii) The analysis of the first stages of the fully coupled non-linear method and (iii) The analysis of the first stages of the semi-uncoupled non-linear method. It may be seen that the results obtained by the three methods are very similar being the discrepancies associated to a slight shift of the vibration frequencies between the non-linear and the linear method and the non-linearity itself. The main conclusion is that the aerodynamics of the system is linear since the fully non-linear time marching method provides nearly the same results than the frequency domain linear methodology.
The post-processing of the time signals of any of the non-linear methods requires the post-processing techniques previously outlined, otherwise the mixing of the forward and backward traveling-waves would prevent a correct derivation of the decay or growth rates of the modes. The problem is specially difficult for the stable modes since they decay very quickly and their associated amplitudes are very small and difficult to separate from the rest. Still the result is quite satisfactory.

The results depicted in Fig. 10 are not fully satisfactory since they do not allow to conclude which is the final state of the system. Our conjecture, that we have already used in previous works [13, 2], is that the final state of a bladed-disk system with multiple unstable modes of the same family in the absence of a periodic forcing, includes just the most unstable mode. Because of the limited amount of simulation time the end time of Fig. 10 corresponds approximately to 150 periods.

The semi-uncoupled method has been used to resume the simulation for very long times, of the order of several thousands of periods. The results may be seen in Fig. 11. It may be appreciated than for very long times just a single TW survives which, incidentally, was the one predicted as most unstable by the linear method (the forward TW of the 7th nodal diameter). This simulation, together with the one performed for the six-sector computational domain, confirms that the conjecture is true. The initial stages of the fully-non linear simulation are shown on the bottom of the same figure. It is seen that the dynamics is the same than that of the semi-uncoupled approach and the non-linearity is filtering out the solution. There are clear indications that only a single model will survive in the long term. This simulation may be seen as a final indication that the conjecture that we formulated is true.

It must be taken into account that the semi-uncoupled method is several orders of magnitude faster than the fully coupled approach. In table 2 the huge difference in the computational cost may be seen. Notice that the most expensive part of the pre-processing for the semi-uncoupled methodology are the linear simulations to obtain the matrix $D(\omega_0)$. Parallel processing may be used to reduce the computational time for the fully coupled approach; in this particular case 16 processors were used with more than 80% of parallel efficiency.

<table>
<thead>
<tr>
<th>Method</th>
<th>Pre-processing</th>
<th>Period Cost</th>
<th>Total cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>Semi-uncoupled</td>
<td>4 hour</td>
<td>0.05 seg</td>
<td>4 hour</td>
</tr>
<tr>
<td>Fully-coupled</td>
<td>1 hour</td>
<td>18 hour</td>
<td>2 years</td>
</tr>
</tbody>
</table>

CONCLUDING REMARKS

A canonical simple non-linear aeroelastic model which is representative of a LPT welded-in-pair rotor blade has been introduced to increase the understanding of unstable bladed-disks with multiple unstable modes of the same family.
haviour we have had to resort to a simplified non-linear solver that has been previously validated against a fully coupled fully non-linear solver to reduce the computational time. The method is conceptual and can only be used to predict the final vibration amplitude of the converged state if it is properly calibrated. The method may be used to perform conceptual studies retaining the flexibility of the disk but it may not be used as it is to investigate the interaction among different families of blade modes.

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