REDUCED MODELING OF TRANSITIONAL EXACT COHERENT STATES IN SHEAR FLOW

Cédric Beaume[1], Greg P. Chini [2], Keith Julien [3] & Edgar Knobloch [4]
[1]Department of Aeronautics, Imperial College London, London SW7 2AZ, UK
[2]Department of Mechanical Engineering & Program in Integrated Applied Mathematics, University of New Hampshire, Durham, NH 03824, USA
[3]Department of Applied Mathematics, University of Colorado at Boulder, Boulder, CO 80309, USA
[4]Department of Physics, University of California, Berkeley, CA 94720, USA

Abstract In parallel shear flows, the lower branch solution follows simple streamwise dynamics. A decomposition of this solution into Fourier modes in this direction yields modes whose amplitudes scale with inverse powers of the Reynolds number. We use this scaling to derive a reduced model for exact coherent structures in general parallel shear flows. The reduced model is regularized by retaining higher order viscous terms. Both lower branch and upper branch solutions are captured and studied.

INTRODUCTION

Transitional parallel shear flows are characterised by the importance of exact coherent structures (ECS). These ECS are formed at low values of the Reynolds number Re through saddle-node bifurcations and are only weakly unstable in the transitional regime. As a result of the small number of unstable eigendirections, the ECS attract transitional flow trajectories which can then be thought of as bouncing from ECS to ECS [3, 6]. Of these, two specific ECS are of particular interest in small domains: the least unstable lower branch solution and the associated upper branch solution. The former is an attractor on the separatrix between the stable laminar unidirectional flow and the attracting turbulent state [8], whereas the latter reproduces important low-order statistics of turbulence [9, 5]. Systematic study of these ECS is not trivial due to their three-dimensionality, the fact that they are not connected to the laminar unidirectional state and that they are unstable. We describe here a more efficient way to compute such ECS and shed light on their structure.

METHODOLOGY

Using a Fourier decomposition in the streamwise direction, Wang *et al.* [10] observed that the lower branch ECS in plane Couette flow are comprised of $O(1)$ streamwise streaks, $O(Re^{-1})$ streamwise rolls and fluctuations and negligibly smaller remaining modes. In the limit of infinite Re, ECS are characterised by the presence of a critical layer, a curve on which fluctuation gradients accumulate [7]. We consider Waleffe flow: a flow confined between two non-moving free-slip parallel walls and driven by a sinusoidal force as sketched in figure 1. We make use of Wang *et al.* scaling [10] to derive

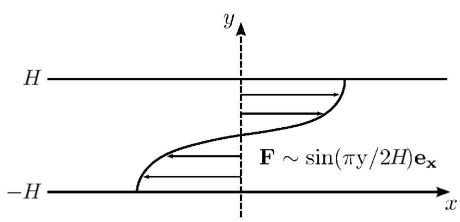

Figure 1. Sketch of Waleffe flow, a cousin of plane Couette flow.

the following set of two-dimensional (y, z) equations for the streaks u_0 and rolls (ω_1,ϕ_1):

$$\partial_T u_0 + J(\phi_1, u_0) = \nabla_\perp^2 u_0 + \frac{\sqrt{2}\pi^2}{4}\sin(\pi y/2), \tag{1}$$
$$\partial_T \omega_1 + J(\phi_1, \omega_1) + 2(\partial_{yy}^2 - \partial_{zz}^2)\left(\mathcal{R}(v_1 w_1^*)\right) + 2\partial_y\partial_z(w_1 w_1^* - v_1 v_1^*) = \nabla_\perp^2 \omega_1, \tag{2}$$

and for the fluctuations in the wall-normal and spanwise (complex) velocity (v_1, w_1) coupled to the pressure fluctuation p_1:

$$(\alpha^2 - \nabla_\perp^2)p_1 = 2i\alpha(v_1\partial_y u_0 + w_1\partial_z u_0), \tag{3}$$
$$\partial_t \mathbf{v}_{1\perp} + i\alpha u_0 \mathbf{v}_{1\perp} = -\nabla_\perp p_1 + \epsilon\nabla_\perp^2 \mathbf{v}_{1\perp}. \tag{4}$$

In writing these, we introduced the long timescale $T = \epsilon t$, and the notation $J(f, g) = \partial_y f \partial_z g - \partial_z f \partial_y g$, $\nabla_\perp = \partial_y^2 + \partial_z^2$, $\mathcal{R}$ for the real part, a star superscript for complex conjugation and α for the wavenumber in the streamwise direction (the fluctuations are expanded as follows: $\mathbf{v}'_{1\perp}(x,y,z,t) = \mathbf{v}_{1\perp}(y,z,t)e^{i\alpha x} + c.c.$). This model is directly obtained after expanding the velocity and pressure fields as in Wang *et al.* [10] and assuming that Re is large; the fluctuation equations (3) and (4) are linear and have been regularised by the retention of the subdominant viscous term $\epsilon\nabla_\perp^2\mathbf{v}_{1\perp}$ in equation (4), where $\epsilon \equiv Re^{-1}$. This term allows the critical layer to have a finite width and form a critical region in accord with numerical observations. In contrast, [4] carry out similar calculations at infinite Reynolds numbers, thereby assuming an infinitely thin critical region. The system (1)–(4) is posed on a two-dimensional domain (y,z) of nondimensional extent L_y, L_z, and involves a unit effective Reynolds number for the mean equations (1) and (2). We take advantage of the structure of the resulting problem by approximating ECS using a two-step algorithm consisting of solving for the fluctuations at fixed streak velocity u_0 using an eigenvalue formulation for their temporal growth rate and then using the obtained fluctuation mode to evolve the streaks and rolls to a steady state, and back again. Once a good approximate ECS is obtained, it is used within a preconditioned Newton algorithm to converge to a decent accuracy. Details on the method are given in [2, 1].

RESULTS

The above method allows an easy calculation of the lower branch ECS and a continuation package has been coded on top of the Newton method to allow continuation in Re. This way, we have continued the lower branch down to its saddle-node and obtained the corresponding upper branch. The lower and upper branch states for $L_y = 2$, $L_z = \pi$ are shown in figure 2 side by side to highlight structural differences. The lower branch ECS (figure 2(a)) is characterised by a critical layer

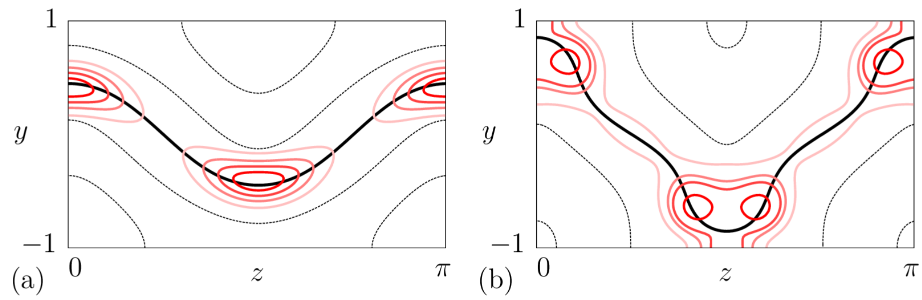

Figure 2. Representation of the streaks u_0 in black lines and of the amplitude of the fluctuations in red for the lower (a) and upper (b) branch states at $Re \approx 1500$. The thick black line represents the critical layer ($u_0 = 0$), the dashed lines represent equispaced contours. Fluctuation amplitude contours are also equispaced.

mildly deformed by quasi-circular rolls (not shown) while the fluctuations are strongly focused around the critical layer and vary slowly along it. In contrast, the critical layer of the upper branch ECS (figure 2(b)) is highly deformed as a result of stronger and more complex rolls. In addition to displaying strong variations perpendicular to the critical layer, the fluctations of the upper branch ECS display important variations along the critical layer. The critical region of the upper branch ECS thus possesses a more complex structure: critical spots are present at the extrema of the critical layer where strong fluctuation gradients are observed in both directions. These results may shed light on structures observed in low Reynolds number turbulence.

References

[1] C. Beaume, G. P. Chini, K. Julien, and E. Knobloch. Reduced description of exact coherent states in parallel shear flows. *submitted*, 2014.
[2] C. Beaume, E. Knobloch, G. P. Chini, and K. Julien. Exact coherent structures in an asymptotically reduced description of parallel shear flows. *Fluid Dyn. Res.*, **47**:015504, 2015.
[3] J. F. Gibson, J. Halcrow, and P. Cvitanović. Equilibrium and travelling-wave solutions of plane couette flow. *J. Fluid Mech.*, **638**:243–266, 2009.
[4] P. Hall and S. Sherwin. Streamwise vortices in shear flows: harbingers of transition and the skeleton of coherent structures. *J. Fluid Mech.*, **661**:178–205, 2010.
[5] G. Kawahara and S. Kida. Periodic motion embedded in plane couette turbulence: regeneration cycle and burst. *J. Fluid Mech.*, **449**:291–300, 2001.
[6] G. Kawahara, M. Uhlmann, and L. van Veen. The significance of simple invariant solutions in turbulent flows. *Annu. Rev. Fluid Mech.*, **44**:203–225, 2012.
[7] S. A. Maslowe. Critical layers in shear flows. *Annu. Rev. Fluid Mech.*, **18**:405–432, 1986.
[8] T. M. Schneider, J. F. Gibson, M. Lagha, F. de Lillo, and B. Eckhardt. Laminar-turbulent boundary in plane couette flow. *Phys. Rev. E*, **78**:037301, 2008.
[9] F. Waleffe. Exact coherent structures in channel flow. *J. Fluid Mech.*, **435**:93–102, 2001.
[10] J. Wang, J. Gibson, and F. Waleffe. Lower branch coherent states in shear flows: transition and control. *Phys. Rev. Lett.*, **98**:204501, 2007.