MUTUAL-FRICTION COEFFICIENTS IN TWO-DIMENSIONAL SUPERFLUIDS: FROM THE GROSS-PITAEVSKII EQUATION TO THE HALL-VINEN-BEKHAREVICH-KHALATNIKOV TWO-FLUID MODEL

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Abstract We start from the two-dimensional Gross-Pitaevskii equation (GPE) and develop algorithms for the ab-initio determination of the temperature (T) dependence of the mutual-friction coefficients, α and α', and the normal-fluid density ρ_n, which appear as parameters in the Hall-Vinen-Bekharevich-Khalatnikov (HVBK) two-fluid model for a superfluid. In the second part of our study, we elucidate the statistical properties of two-dimensional, homogeneous, isotropic superfluid turbulence in the simplified HVBK model, with values for the mutual-friction coefficients that are comparable to those we obtain from the first part of our study.

INTRODUCTION

Theoretical treatments of superfluid turbulence use a variety of models, which are applicable at different length scales and for different interaction strengths. At low temperatures T and for weakly interacting bosons, the Gross-Pitaevskii (GP) equation provides a good hydrodynamical description of a superfluid with quantum vortices. If we consider length scales that are larger than the mean separation between quantum vortices, and if we concentrate on low-Mach-number flows, then the two-fluid model of Hall, Vinen, Bekharevich, and Khalatnikov (HVBK) provides a good description of superfluid turbulence. In the HVBK equations, the normal and superfluid velocities are coupled by two mutual-friction coefficients, α and α'. The determination of $\alpha(T)$, $\alpha'(T)$, and the normal-fluid density $\rho_{\rm n}(T)$ from experiments and a combination of analytical and numerical methods, is a challenging problem (see Ref. [1] and references therein).
In the HVBK model, a superfluid vortex does not move with the superfluid velocity $\mathbf{v}_{\rm s}$ but with velocity

$$\mathbf{v} = \mathbf{v}_{\rm sl} + \alpha \mathbf{s}' \times (\mathbf{v}_{\rm n} - \mathbf{v}_{\rm sl}) - \alpha' \mathbf{s}' \times [\mathbf{s}' \times (\mathbf{v}_{\rm n} - \mathbf{v}_{\rm sl})], \tag{1}$$

where $\mathbf{v}_{\rm sl} = \mathbf{v}_{\rm s} + \mathbf{v}_{\rm si}$ is the local superfluid velocity, with $\mathbf{v}_{\rm s}$ and $\mathbf{v}_{\rm si}$ the imposed superfluid velocity and the self-induced velocity because of the vortices, respectively, and $\mathbf{s}'$ the unit tangent at a point on the vortex, with position vector $\mathbf{s}$. Our algorithm for the determination of the mutual-friction coefficients is based on the examinations of the spatiotemporal evolutions of the following two initial configurations in the 2D Galerkin-truncated GP equation (TGPE): (1) $\psi_{\tt IC1}$; and (2) $\psi_{\tt IC2}$. $\psi_{\tt IC1}$ corresponds to a small, vortex-antivortex pair translating with a constant velocity along the x direction at a finite temperature. $\psi_{\tt IC2}$ corresponds to a vortex lattice (by virtue of the periodic boundary conditions), in which we place vortices of alternating signs on the corners of a square at a finite temperature in the presence of a counterflow. We use $\psi_{\tt IC1}$ to determine $\alpha(T)$ and $\psi_{\tt IC2}$ to calculate both $\alpha(T)$ and $\alpha'(T)$ [1].
In the simplified, incompressible, 2D HVBK two-fluid model, the mutual-friction terms can be written as $\mathbf{F}^{\rm n}_{\rm mf} = (\rho_s/\rho)\mathbf{f}_{\rm mf}$ and $\mathbf{F}^{\rm s}_{\rm mf} = -(\rho_n/\rho)\mathbf{f}_{\rm mf}$, where ρ_n/ρ (ρ_s/ρ) is the normal-fluid (superfluid) density fraction; $\mathbf{f}_{\rm mf} = \frac{B}{2}\frac{\omega_{\rm s}}{|\omega_{\rm s}|} \times (\omega_{\rm s} \times \mathbf{u}_{\rm ns}) + \frac{B'}{2}\omega_{\rm s} \times \mathbf{u}_{\rm ns}$, with $\mathbf{u}_{\rm ns} = (\mathbf{u}_{\rm n} - \mathbf{u}_{\rm s})$ the slip velocity, and $B = 2\alpha/\frac{\rho_n}{\rho}$ and $B' = 2\alpha'/\frac{\rho_n}{\rho}$ the coefficients of mutual friction. In most of our studies we set $B' = 0$ so, in 2D, $\mathbf{f}_{\rm mf} = -\frac{B}{2}|\omega_{\rm s}|\mathbf{u}_{\rm ns}$ (see Ref. [2] for details and references therein for 3D studies).

RESULTS

We use direct numerical simulations (DNSs) of the 2D TGPE to show that the determination of $\alpha(T)$ and $\alpha'(T)$ is far more challenging in 2D than it is in three dimensions (3D) because of large fluctuations. In Fig. 1(a), we plot, versus the scaled temperature $T/\tilde{T}_{\rm BKT}$, where $\tilde{T}_{\rm BKT}$ is a rough, energy-entropy-argument estimate of the Berezinskii-Kosterlitz-Thouless (BKT) transition temperature, $\rho_{\rm n}$ (green curve), $(1-\rho_{\rm n})$ (sky-blue curve), and the condensate fraction N_0/N (purple line), where N_0 is the population of the zero-wave-number mode. Figure 1(b) shows the temperature dependence of $\alpha(T)$ and $B = 2\alpha/\frac{\rho_n}{\rho}$, determined using the initial configurations $\psi_{\tt IC1}$ and $\psi_{\tt IC2}$.
In the second part of our study, we use values for the mutual-friction coefficients, which are comparable to those we obtain from the first part of our study, in the DNSs of 2D HVBK equations, which we have designed to study the statistical properties of inverse and forward cascades. We find the following: (1) Both normal-fluid and superfluid energy spectra, $E^n(k)$ and $E^s(k)$, respectively, show inverse- and forward-cascade power-law regimes (Fig. 2(a)). (2) The forward-cascade power law depends on (a) the friction coefficient, as in 2D fluid turbulence, and, in addition, on (b) the coefficient

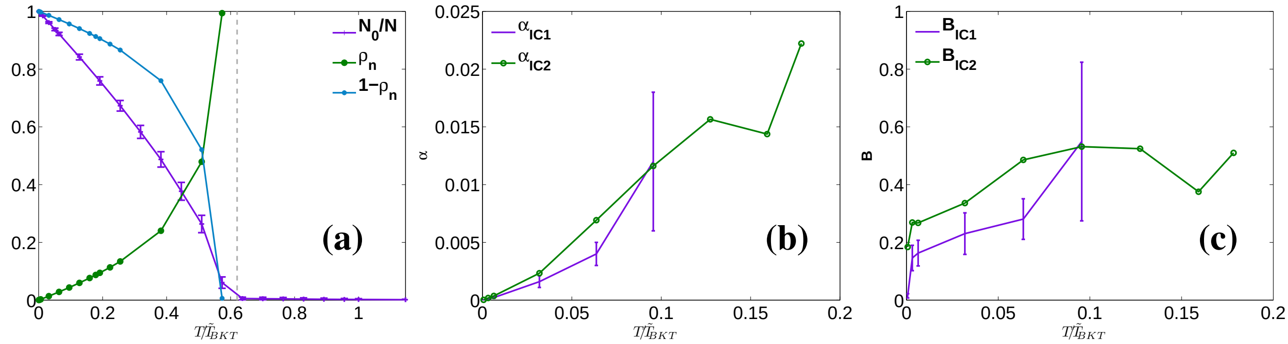

Figure 1. (a) The condensate fraction N_0/N (purple line), the normal fluid density $\rho_{\rm n}$ (green line), and $1-\rho_{\rm n}$ (sky-blue line) versus $T/\tilde{T}_{BKT}$; (c) the mutual friction coefficients $\alpha_{\tt IC1}$ (purple line) and $\alpha_{\tt IC2}$ (green line) versus $T/\tilde{T}_{BKT}$; (d) $B = 2\alpha/\frac{\rho_n}{\rho}$ versus $T/\tilde{T}_{BKT}$.

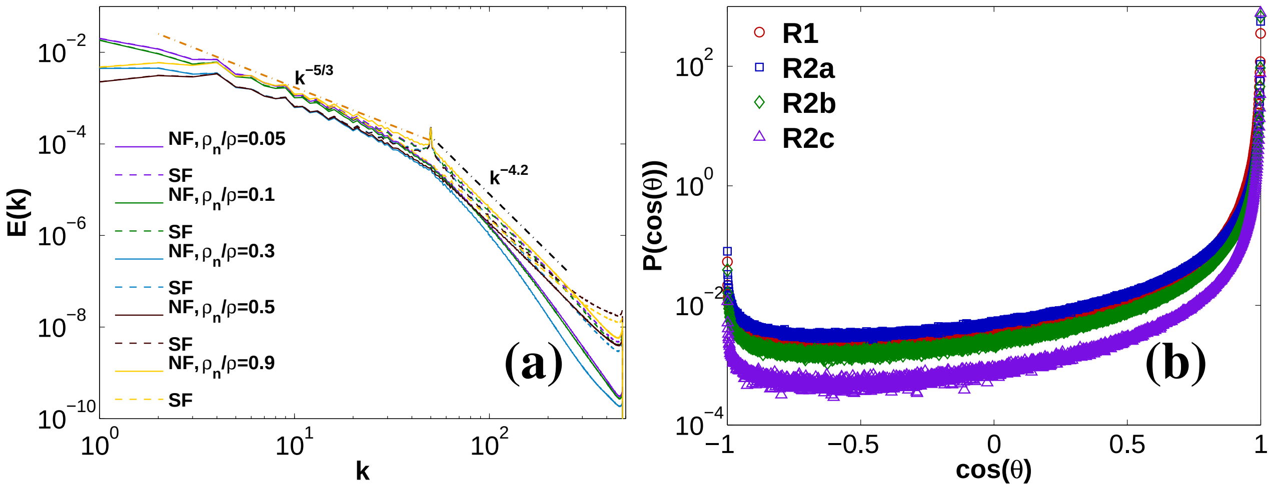

Figure 2. (a) Log-log plots of the energy spectra $E_n(k)$ (full lines) and $E_s(k)$ (dashed lines) from our DNS runs; inverse and forward cascades are shown for different ratios of ρ_n/ρ (the abbreviation NF (SF) stands for normal-fluid (superfluid)). (b) Semilogarithmic (base 10) plots of the PDF $P(\cos(\theta))$ of the angle θ between $\mathbf{u}_{\rm n}$ and $\mathbf{u}_{\rm s}$ for runs `R1` (red circles), `R2a` ($B=1$, blue squares), `R2b` ($B=2$, green diamonds), and `R2c` ($B=5$, purple triangles). For the DNS run `R1`, the external forcing wave number $k_f = 2$; for all other DNS runs shown in the two plots $k_f = 50$. (See Ref. [2] for run parameters.)

B of mutual friction, which couples normal and superfluid velocities. (3) As B increases, the normal and superfluid velocities, $\mathbf{u}_n$ and $\mathbf{u}_s$, respectively, tend to get locked to each other, and, therefore, $E^s(k) \simeq E^n(k)$. (4) We quantify this locking tendency by calculating the probability distribution functions (PDFs) $P(\cos(\theta))$ and $P(\gamma)$, where the angle $\theta \equiv \cos^{-1}((\mathbf{u}_{\rm n}\cdot\mathbf{u}_{\rm s})/(|\mathbf{u}_{\rm n}||\mathbf{u}_{\rm s}|))$ and the amplitude ratio $\gamma = |\mathbf{u}_{\rm n}|/|\mathbf{u}_{\rm s}|$; the former has a peak at $\cos(\theta)=1$ (Fig. 2(b)); and the latter exhibits a peak at $\gamma = 1$ and power-law tails on both sides of this peak. (5) This locking increases as we increase B, but the power-law exponents for the tails of $P(\gamma)$ are universal, in so far as they do not depend on B, ρ_n/ρ, and the forcing wave number $k_{\rm f}$.

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