AN ANALYTICAL CRITERION FOR CENTRIFUGAL INSTABILITY IN NON-AXISYMMETRIC VORTICES

David Nagarathinam[1] , A. Sameen [1] & Manikandan Mathur [1]
[1]*Department of Aerospace Engineering, IIT Madras, Chennai - 600 036, India*

Abstract Non-axisymmetric vortices are ubiquitous in nature; examples include polar vortices in planets, the giant red spot in Jupiter, tornadoes and cyclones on Earth, mesoscale eddies in the ocean. Turbulent flows are furthermore known to be dominated by small- and large-scale vortex structures. Owing to the wide range of applications, knowledge of conditions under which a given vortex becomes unstable is beneficial. Here, the centrifugal instability of two-dimensional, non-axisymmetric vortices in the presence of an axial flow (w) and a background rotation (Ω_z) is studied using the local stability approach. The local stability approach, based on geometric optics and similar in formulation to the rapid distortion theory [2], considers the evolution of shortwavelength perturbations along streamlines in the base flow. This approach, developed by Lifschitz & Hameiri [4], is particularly useful for base flows for which a global stability analysis is computationally expensive. A sufficient criterion for centrifugal instability in an axisymmetric vortex with (w) and (Ω_z) is first derived by analytically solving the local stability equations for wave vectors that are periodic upon evolution around a closed streamline. This criterion is then heuristically extended to non-axisymmetric vortices and written in terms of integral quantities on a streamline. The criterion is then shown to be accurate in describing centrifugal instability over a reasonably large range of parameters that specify Stuart vortices and Taylor-Green vortices.

The local stability approach investigates the evolution of short-wavelength perturbations in velocity $\mathbf{u}$ and pressure p, assumed to be of the form:

$$(\mathbf{u}, p) = \exp\Big(i\frac{\phi(\mathbf{x},t)}{\epsilon}\Big)\Big[\big(\mathbf{a}(\mathbf{x},t), \pi(\mathbf{x},t)\big) + \epsilon\big(\mathbf{a}_\epsilon(\mathbf{x},t), \epsilon\pi_\epsilon(\mathbf{x},t)\big) + ...\Big], \tag{1}$$

where ϕ is a real scalar function of position $\mathbf{x}$ and time t, ϵ a small parameter and $\mathbf{k} = \nabla\phi$ the wave vector. The wave vector and the leading order complex amplitudes ($\mathbf{a}$ & π) in an incompressible, inviscid flow are governed by [2]:

$$\frac{\mathrm{d}\mathbf{k}}{\mathrm{d}t} = [(\nabla \times \mathbf{U_B}) \times \mathbf{k} - (\mathbf{k}\cdot\nabla)\mathbf{U_B}], \tag{2}$$

$$\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t} = -\nabla\mathbf{U_B}\cdot\mathbf{a} + \frac{2}{|\mathbf{k}|^2}[(\nabla\mathbf{U_B}\cdot\mathbf{a})\cdot\mathbf{k}]\mathbf{k} - 2\mathbf{\Omega_B}\times\mathbf{a} + \frac{2}{|\mathbf{k}|^2}[(\mathbf{\Omega_B}\times\mathbf{a})\cdot\mathbf{k}]\mathbf{k}, \tag{3}$$

with $\pi = 0$ and $\mathbf{k}\cdot\mathbf{a} = 0$. Here, $\mathbf{U_B}$ is the base flow velocity field, $\mathbf{\Omega_B}$ the background rotation and $\mathrm{d}/\mathrm{d}t = \partial/\partial t + \mathbf{U_B}\cdot\nabla$ the material time derivative in the base flow $\mathbf{U_B}$, i.e. derivative along fluid trajectories in the base flow.

We start by considering a steady, axisymmetric vortex (in the xy-plane) described by the velocity $\mathbf{U_B} = \psi'(r)\mathbf{e}_\theta + w(r)\mathbf{e_z}$ with a background rotation $\mathbf{\Omega_B} = \Omega_z\mathbf{e_z}$, where $\mathbf{e_r}$, $\mathbf{e}_\theta$, $\mathbf{e_z}$ are the unit vectors in the cylindrical polar coordinates. Invoking the periodicity criterion [5] for unit wave vectors along a closed streamline, the instability criterion is shown to be:

$$(\sigma_C^*)^2 = \Big[\frac{4w'^2(r\Omega_z + \psi')^2}{(r\psi'' - \psi')^2 + r^2w'^2} - 2\Big(\frac{\psi'}{r} + \Omega_z\Big)\Big(\psi'' + \frac{\psi'}{r} + 2\Omega_z\Big)\Big] > 0, \tag{4}$$

which converges to Rayleigh's criterion [8] in the limit of zero axial flow ($w = 0$) and no background rotation ($\Omega_z = 0$). In the absence of background rotation, it reduces to the criterion given in [3] & [5]. In the absence of axial flow, the criterion reduces to the criteria in [6] & [9].

Recognizing the circulation Γ and time period T as dynamically significant parameters on a streamline, the growth rate for any non-axisymmetric vortex is calculated by replacing $\mathrm{d}/\mathrm{d}r$ by $\psi'\mathrm{d}/\mathrm{d}\psi$, $r\psi'$ by $\Gamma/2\pi$ and ψ'/r by $2\pi/T$ in the expression (4). The centrifugal instability criterion in non-axisymmetric vortices is then written as:

$${\sigma_C^*}^2 = \frac{4(dw/d\psi)^2(\Omega_z + 2\pi/T)^2}{(\Gamma/T^3)(\frac{dT}{d\psi})^2 + (dw/d\psi)^2} - 2\Big(\frac{2\pi}{T} + \Omega_z\Big)\Big(\frac{1}{T}\frac{d\Gamma}{d\psi} + 2\Omega_z\Big) > 0. \tag{5}$$

The above criterion converges to that of Bayly [1] for $w = 0$ and $\Omega_z = 0$, and to the heuristic criterion of Mathur *et. al.*[5] when $\Omega_z = 0$.

In this talk, we will discuss the validity of criterion (5) in describing the centrifugal instability in two different models of non-axisymmetric vortices, namely the Stuart vortices and the Taylor-Green vortex. We also discuss how our new

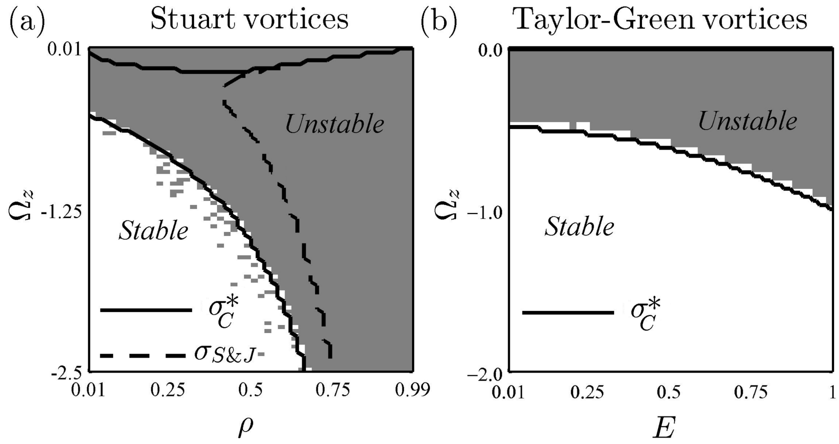

Figure 1. Contours delineating the stable and unstable flow regimes in Stuart vortices (left) and Taylor-Green vortices (right) without an axial flow ($w = 0$). The unstable domain in the (ρ, Ω_z) and (E, Ω_z) parameter space evaluated based on our criterion (5) is bound within the solid lines. The unstable domain in the (ρ, Ω_z) parameter space evaluated based on Sipp & Jacquin's criterion [9] is bound within dotted lines. Grey background indicates unstable domains captured by the numerical solutions of the local stability equations.

criterion, apart from being more accurate than that of Sipp & Jacquin [9] in the limit of $w = 0$, describes centrifugal instability reasonably accurately in the presence of both axial flow and background rotation. The validity of our criterion is investigated by comparing σ_C^* from criterion (5) with the growth rate calculated by numerically solving equations (2) - (3).

The stream function describing Stuart vortices in the xy-plane is given by:

$$\psi(x, y) = \log(\cosh y - \rho \cos x), \tag{6}$$

where ρ is the concentration parameter. An isolated, steady, flattened Taylor-Green vortex with center at the origin is given by the following stream function:

$$\psi = \sin(x - \pi/2)\sin(Ey + \pi/2), \tag{7}$$

where E is the aspect ratio of the vortex. The distance of both ρ and E from unity is a measure of the extent of non-axisymmetry of the respective vortices.

As shown in figure 1, the unstable domain (in the $\rho - \Omega_z$ plane for Stuart vortices and the $E - \Omega_z$ plane for the Taylor-Green vortex) predicted by criterion (5) is remarkably close to the overall unstable domain captured by the numerical solutions of the local stability equations. Furthermore, the criterion based on Sipp & Jacquin [9] is less accurate for Stuart vortices, and fails to identify any unstable pair (E, Ω_z) for the Taylor-Green vortex.

In this talk, we will present detailed comparisons between criterion (5) and numerical solutions for all the streamlines corresponding to several individual cases of (ρ, Ω_z) and (E, Ω_z). The influence of the axial flow on centrifugal instability will then be discussed, followed by an analytical estimate of the threshold magnitude of anticyclonic background rotation above which non-axisymmetric vortices are centrifugally stable. More details on the present work can be found in [7].

References

[1] B. J. Bayly. Three-dimensional centrifugal-type instabilities in inviscid two-dimensional flows. *Physics of Fluids* **31**: 56–64, 1998.

[2] F. S. Godeferd, C. Cambon, and S. Leblanc. Zonal approach to centrifugal, elliptic and hyperbolic instabilities in Stuart vortices with external rotation. *Journal of Fluid Mechanics* **449**: 1–37, 2001.

[3] S. Leibovich and K. Stewartson. A sufficient condition for the instability of columnar vortices. *Journal of Fluid Mechanics* **126**: 335–356, 1983.

[4] A. Lifschitz, E. Hameiri. Local stability conditions in fluid dynamics. *Physics of Fluids A* **3**: 2644–2651, 1991.

[5] M. Mathur, S. Ortiz, T. Dubos and J. M. Chomaz. Effects of an axial flow on the centrifugal, elliptic and hyperbolic instabilities in Stuart vortices. *Journal of Fluid Mechanics* **758**: 565–585, 2014.

[6] I. Mutabazi, C. Normand and J. E. Wesfreid. Gap size effects on centrifugally and rotationally driven instabilities. *Physics of Fluids A* **4**: 1199–1205, 1992.

[7] D. Nagarathinam, M. Mathur, and A. Sameen. Centrifugal instability in non-axisymmetric vortices *Journal of Fluid Mechanics* **Submitted**, 2014.

[8] O. M. Lord Rayleigh. On the dynamics of revolving fluids. *Proc. R. Soc. Lond. A.* **93**: 148–154, 1917.

[9] D. Sipp and L. Jacquin. Three-dimensional centrifugal-type instabilities of two-dimensional flows in rotating systems. *Physics of Fluids* **12**: 1740–1748, 2000.