AN ALTERNATIVE DEFINITION OF ORDER DEPENDENT DISSIPATION SCALES

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Abstract While Kolmogorov's similarity hypothesis suggests that velocity structure functions scale with the mean dissipation $\langle\varepsilon\rangle$ and the viscosity ν, we find that the $2m$. even order scales with $\langle\varepsilon^m\rangle$. This implies that there are other cut-off lengths than the Kolmogorov length η. These cut-off lengths are smaller than η and decrease with increasing order and Reynolds-number. They are compared to a previous definition of order dependent dissipative scales by Schumacher et. al[4].

Although the governing equations for incompressible turbulent flows, the Navier-Stokes equations, are known for quite some time, it is not possible to solve them analytically. For that reason, statistical methods are applied to arrive at a better understanding of turbulent flows. In particular, correlation functions between two points separated by a distance r are of interest, as they describe spatial properties of the flow. As turbulence is a multi-scale problem, correlation functions and similar constructs are also suited to examine the properties of the flow at different scales. Kolmogorov[3] proposed two similarity laws, namely that the statistics of structure functions (the velocity difference between two points separated by a distance r) are determined by the viscosity ν and the mean dissipation $\langle\varepsilon\rangle$ for locally isotropic turbulence for small r (first hypothesis of similarity), while for r situated between the very small scales and the large scales the dependence on the viscosity ν should vanish (second hypothesis of similarity). From the two quantities ν and $\langle\varepsilon\rangle$ relevant at the very small scales, he introduced $\eta = (\nu/\langle\varepsilon\rangle)^{1/4}$ and $u_\eta = (\nu\langle\varepsilon\rangle)^{1/2}$ as characteristic length scale and velocity, inasmuch as the second order structure function (the square of the velocity difference) should be completely determined by ν, $\langle\varepsilon\rangle$ and a (unknown) function $f(r/\eta)$ for all r. In a second paper[2], he proceeded to rewrite the Kárman-Howarth equation in terms of the second order longitudinal structure function. This allowed him to give analytic solutions for the second order structure function for $r \to 0$ and the third order structure function in the inertial range $\eta \ll r \ll L$ (under the assumption of very large (infinite) Reynolds-number), where L is the inertial length scale.
Expanding the second order $D_{20} = \langle (u_1(x_i + r_i) - u_1(x_i))^2 \rangle$ for $r \to 0$ gives

$$D_{20} = \frac{1}{2}\left\langle \left(\frac{\partial u_1}{\partial x_1}\right)^2 \right\rangle r^2. \tag{1}$$

Normalising with η and u_η^2 yields

$$\frac{D_{20}}{u_\eta^2} = \frac{1}{15}\left(\frac{r}{\eta}\right)^2, \tag{2}$$

where $\langle\varepsilon\rangle = 15\nu\langle(\partial u_1/\partial x_1)^2\rangle$ due to isotropy has been used. Thus, the second order structure function collapses if normalised with η and u_η for all Reynolds-numbers in the dissipative range.
However, the fourth order D_{40} is determined by $\langle(\partial u_1/\partial x_1)^4\rangle$ (for $r \to 0$) which can not be expressed in terms of $\langle\varepsilon\rangle^2$ as the first similarity hypothesis would suggest. Rather, we find that D_{40} is collapsed by $\langle\varepsilon^2\rangle$, D_{60} by $\langle\varepsilon^3\rangle$ and so on. This implies that higher orders are cut off at different length scales, namely

$$\eta_{C,2m} = \left(\frac{\nu^3}{\langle\varepsilon^m\rangle^{1/m}}\right)^{1/4} \tag{3}$$

with velocity

$$u_{C,2m} = \left(\nu\langle\varepsilon^m\rangle^{1/m}\right)^{1/4}. \tag{4}$$

It follows from eq. (3) that

$$\frac{\eta_{C,2m}}{\eta} = \left(\frac{\langle\varepsilon\rangle^m}{\langle\varepsilon^m\rangle}\right)^{1/4m} \sim Re_\lambda^{-\frac{\alpha(m)}{4m}}, \tag{5}$$

where $\langle\varepsilon^m\rangle/\langle\varepsilon\rangle^m \sim Re_\lambda^{\alpha(m)}$ with $\alpha(m+1) > \alpha(m) > 0$. Therefore, $\eta_{C,2m} < \eta$ and that ratio increases with Reynolds-number and order m. Figure 1 shows $\eta_{C,2m}$ for $m = 1, .., 5$ and $Re_\lambda = 88, .., 529$. As expected, we find indeed that $\eta_{C,2m}$ decreases with increasing Reynolds-number and order m.
Under the assumption that velocity increments at large scales follow a Gaussian distribution, Schumacher et. al.[4] derived

$$\eta_{2m} \sim Re_\lambda^{\frac{1}{\zeta_{2m} - \zeta_{2m+1} - 1}} \tag{6}$$

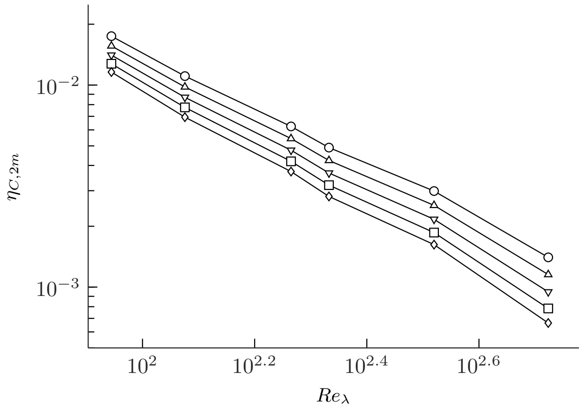

Figure 1. $\eta_{C,2m}$ as function of Re_λ. $\circ$ $m = 1$ (i.e. the Kolmogorov scale η), $\triangle$ $m = 2$, $\triangledown$ $m = 3$, $\square$ $m = 4$ and $\diamond$ $m = 5$.

where L is the integral length scale, Re_L the large scale Reynolds-number and ζ_n the scaling exponent of the longitudinal structure function of order n in the inertial range. Rewriting eq. (6) and using eq. (5) then results in

$$\zeta_{2m+1} - \zeta_{2m} = \left(\frac{1}{2}\frac{\alpha(m)}{4m} + \frac{3}{4}\right)^{-1} - 1. \tag{7}$$

Thus, from the Hölder inequality (cf. Frisch[1]) we find that

$$\frac{\alpha(m)}{4m} \leq \frac{1}{2} \tag{8}$$

and with eq. (5) that $\eta_{C,2m}$ approaches a constant for a given Reynolds-number. Consequently, we find a modified upper limit of the number of required grid points for DNS simulations, i.e.

$$N \sim \left(\frac{L_{Box}}{\Delta x}\right)^3 \sim \left(\frac{L_{Box}}{L}\right)^3 \left(\frac{L}{\eta}\right)^3 \left(\frac{\eta}{\eta_{C,2m}}\right)^3 \lesssim \left(\frac{L_{Box}}{L}\right)^3 Re_L^3 \tag{9}$$

compared to $N \sim Re_L^{9/4}$ for K41 theory.

References

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