DISSIPATIVE RANGE SCALING OF HIGHER ORDER STRUCTURE FUNCTIONS FOR VELOCITY AND PASSIVE SCALARS

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Abstract Differently to Kolmogorov's second similarity hypothesis, we find that the $2n$-th order velocity and scalar structure functions scale with $\langle\varepsilon^n\rangle$ and $\langle\chi^n\rangle$, respectively. The origins of this scaling are analyzed by the transport equations of the fourth order velocity and scalar increment moments and by direct numerical simulations.

Since the seminal work of Kolmogorov [1941a,b] the scaling of structure functions in statistically isotropic and homogeneous turbulence has been of paramount interest. Structure functions can be defined by the moments of the velocity or scalar increments, and read for the velocity increment

$$S_{n,m}(r) = \langle (u_1(x_1+r) - u_1(x_1))^n \, (u_2(x_1+r) - u_2(x_1))^m \rangle \,, \tag{1}$$

where r is the separation distance between two independent points and the angular brackets denote an ensemble-average. Kolmogorov's theory proposes that structure functions follow a power-law scaling relation.

In the dissipative range, for $r \to 0$, the structure functions become analytical and can be expanded as

$$\lim_{r\to 0} \frac{S_{n,m}}{r^{n+m}} \propto \langle \lim_{r\to 0} \left(\frac{\Delta u_1}{r}\right)^n \left(\frac{\Delta u_2}{r}\right)^m \rangle = \langle \left(\frac{\partial u_1}{\partial x_1}\right)^n \left(\frac{\partial u_2}{\partial x_1}\right)^m \rangle \,. \tag{2}$$

Kolmogorov's first similarity hypothesis proposes that statistics in the dissipative range depend solely on the mean dissipation $\langle\varepsilon\rangle$ and on the viscosity ν. However, velocity gradients, as small-scale quantities, exhibit a probability density function with a complex shape and stretched-exponential tails. This tails originate from strong rare events which are non-universal and generally depend on Reynolds number. Higher order moments are mostly determined by the tails of the probability density function. Therefore, it cannot be expected that higher order moments of the velocity derivative can be expressed in terms of $\langle\varepsilon\rangle$. Instead, dimensional analysis suggests that the $2n$-th moments of the velocity gradients scale with $\langle\varepsilon^n\rangle/\nu^n$ rather than with $\langle\varepsilon\rangle^n/\nu^n$. Note, that this relation is not derived from first principles, but it is supported by DNS calculations as shown in tab. 1 where the ratios,

$$\nu^{n+m} \langle \left(\frac{\partial u_1}{\partial x_1}\right)^{2n} \left(\frac{\partial u_2}{\partial x_1}\right)^{2m} \rangle / \langle \varepsilon^{n+m} \rangle \,, \tag{3}$$

are given as a function of the Taylor based Reynolds number for the fourth order. Indeed, the value of this ratio is independent of Reynolds number. The same result is valid for the sixth order. The non-dimensional ratios $\langle\varepsilon^m\rangle/\langle\varepsilon\rangle^m$ are known to depend on Reynolds number, cf. Nelkin [1994], with $\langle\varepsilon^m\rangle/\langle\varepsilon\rangle^m \propto \mathrm{Re}_\lambda^{f(m)}$.

Table 1. Evaluation of eq. 3 from direct numerical simulations for Reynolds numbers between $\mathrm{Re}_\lambda = 88$ and $\mathrm{Re}_\lambda = 754$.

Re_λ	88	119	184	215	331	529	754
$\nu^2\langle\left(\frac{\partial u_1}{\partial x_1}\right)^4\rangle/\langle\varepsilon^2\rangle$	0.0096	0.0096	0.0095	0.0096	0.0095	0.0095	0.0095
$\nu^2\langle\left(\frac{\partial u_1}{\partial x_1}\right)^2\left(\frac{\partial u_2}{\partial x_1}\right)^2\rangle/\langle\varepsilon^2\rangle$	0.0537	0.0551	0.0570	0.0575	0.0586	0.0588	0.0596
$\nu^2\langle\left(\frac{\partial u_2}{\partial x_1}\right)^4\rangle/\langle\varepsilon^2\rangle$	0.0078	0.0078	0.0080	0.0080	0.0080	0.0080	0.0080

Figure 1 shows the fourth and sixth order longitudinal velocity structure functions as well as the fourth and sixth order scalar structure functions for Taylor based Reynolds numbers varying between 88 and 754. The structure functions are compensated by their dissipative range scaling. We find that for the passive scalar structure functions this scaling collapses the curves for all Reynolds numbers in the dissipative and inertial range, indicating that the dissipative range scaling is valid as well in the inertial range. This is different for the velocity structure functions. Here, the compensation by the dissipative range scaling collapses the curves in the dissipative range, but not entirely in the inertial range.

The difference in the scaling between scalar and velocity structure functions is further analyzed by their respective dynamic equations. The transport equation for the fourth order longitudinal velocity structure function reads

$$\frac{\partial S_{4,0}}{\partial t} + \left(\frac{\partial}{\partial r} + \frac{2}{r}\right) S_{5,0} - \frac{8}{r} S_{3,2} = -2\nu E_{4,0} + 2v \left[\frac{\partial^2 S_{4,0}}{\partial r^2} + \frac{2}{r}\frac{\partial S_{4,0}}{\partial r} - \frac{8}{r^2} S_{4,0} + \frac{24}{r^2} S_{2,2}\right] - T_{4,0} \,, \tag{4}$$

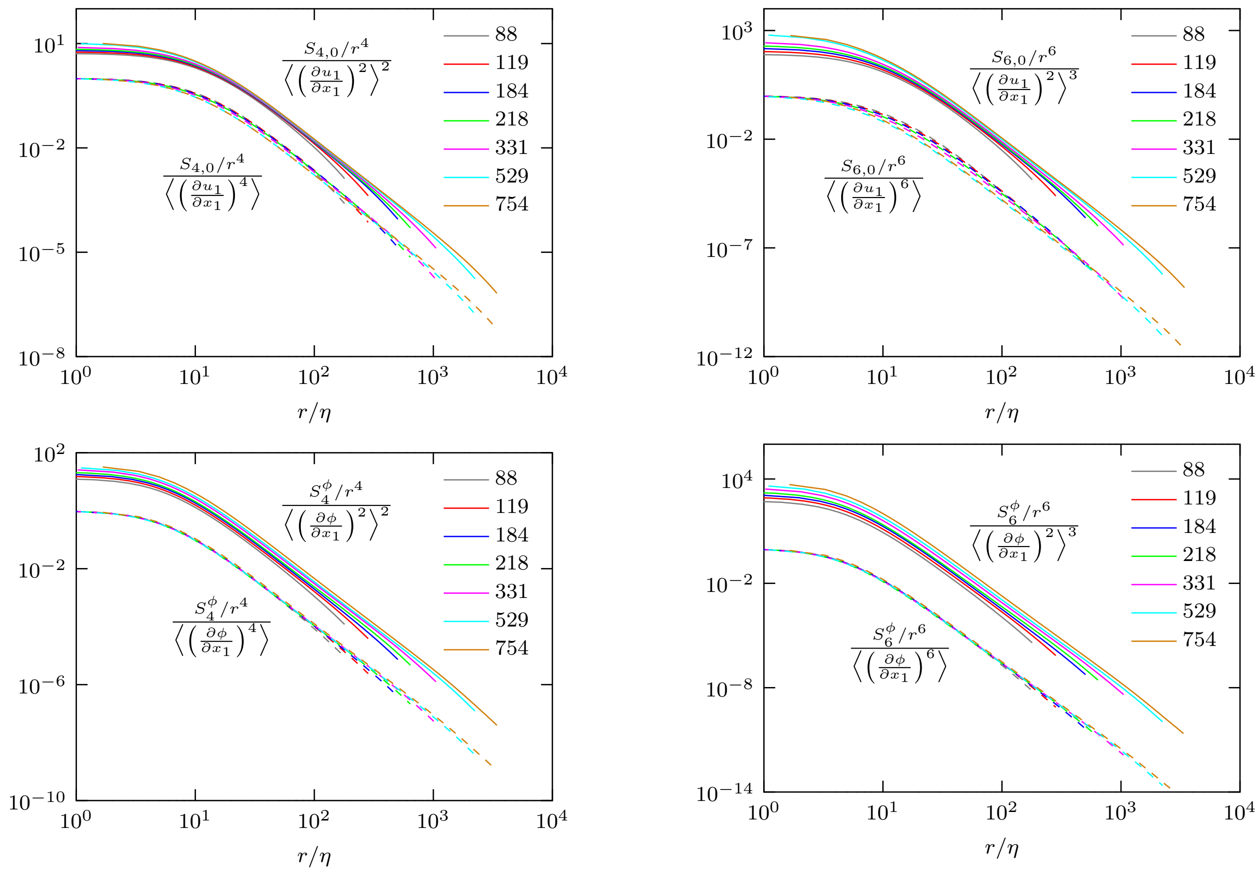

Figure 1. Fourth (left) and sixth (right) order longitudinal velocity (top) and scalar structure functions for Taylor based Reynolds numbers between 88 and 754. The curves are compensated with Kolmogorov's scaling (solid lines) and the dissipative range scaling (dashed lines).

where $T_{4,0}$ is a pressure term and $E_{4,0}$ is a dissipation term, cf. Hill [2014]. The transport equation for the fourth order scalar structure function reads

$$\frac{\partial\langle(\Delta\phi)^4\rangle}{\partial t}+\langle\left(\frac{\partial}{\partial r}+\frac{2}{r}\right)(\Delta u_L)(\Delta\phi)^4\rangle=-12D\langle(\Delta\phi)^2\left(\chi'+\chi\right)\rangle+2D\left[\frac{\partial^2\langle\Delta\phi\rangle^4}{\partial r^2}+\frac{2}{r}\frac{\partial\langle(\Delta\phi)^4\rangle}{\partial r}\right]. \quad (5)$$

Equations 4 and 5 exhibit both the same structure. They describe the balance between a temporal term, a transport term, a dissipation term, and a diffusion term (from left to right). However, eq. 4 additionally comprises the pressure term $T_{4,0}$ which modifies the inertial range scaling, cf. Kurien and Sreenivasan [2001]. Therefore, for the fourth order velocity structure function, the statistics of the dissipation are not sufficient to fully collapse the curves in both dissipative and inertial range. In the dissipative range the pressure term scales as $T_{4,0} \propto r^3$, while the dissipation term scales as $E_{4,0} \propto r^2$. Thus, for $r \to 0$ the dissipation effect is dominant, and the velocity structure functions can be collapsed solely by the moments of the dissipation according to eqs. 2 and 3.

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