HELICAL MODE INTERACTIONS AND SPECTRAL ENERGY TRANSFER IN MAGNETOHYDRODYNAMIC TURBULENCE

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Abstract Spectral transfer processes in magnetohydrodynamic (MHD) turbulence are investigated by decomposition of the velocity and magnetic fields in Fourier space into helical modes. In 1992, Waleffe (Phys. Fluids A, **4** 350 (1992)) used this decomposition to calculate triad interactions for isotropic hydrodynamic turbulence and determined whether a given triad contributed to forward or reverse energy transfer depending on the helicities of the interacting modes. The problem becomes more difficult in MHD due to the need to treat a coupled system of partial differential equations and the energy transfers between the magnetic and velocity fields. This requires the development of techniques that extend Waleffe's work, which are subsequently used to calculate the direction of energy transfer processes originating from triad interactions derived from the MHD equations. In order to illustrate the possible transfer processes that arise from helical mode interactions, we focus on simplified cases and putting special emphasis on interactions resulting in reverse spectral energy transfer. This approach also proves to be helpful in determining the nature of certain energy transfer processes, where transfer of energy between different fields and between the same field can be distinguished. Reverse transfer of magnetic energy was found if the helicities of two modes corresponding to the smaller wavenumbers are the same, while for reverse transfer of kinetic energy Waleffe's result is recovered. Reverse transfer of kinetic to magnetic energy is facilitated if the interacting magnetic field modes are of opposite helicity, and no reverse transfer of magnetic to kinetic energy was found. More generally, the direction of energy transfer not only depends on helicity but also on the ratio of magnetic to kinetic energy. For the magnetically dominated case reverse transfer occurs of all helicities are the same, the kinetically dominated case two modes need to have the same helicity while the third mode is of opposite helicity to allow reverse transfer.
Keywords: magnetohydrodynamics, turbulence, helical mode decomposition, spectral methods

INTRODUCTION

Waleffe [1] showed that the dynamical system that governs a single wavevector triad interaction has certain steady solutions, from a stability analysis of these solutions he was able to derive the direction of spectral kinetic energy transfer depending on the helicities of the interacting velocity field modes. If the helicities of the interacting modes were different, Waleffe obtained a forward kinetic energy transfer from smaller to larger wavenumbers. If the helicities of the modes corresponding to the two larger wavenumbers were the same, then a reverse transfer became possible. This was confirmed numerically by Biferale *et al.* by direct numerical simulations of the Navier-Stokes equations keeping only interactions of helical modes of the same sign [2]. Lessinnes, Plunian and Carati [3] carried the helical mode decomposition over to magnetohydrodynamics and derived a dynamical system describing triadic interactions in MHD, which was subsequently used to construct shell models. In this paper we extend Waleffe's approach and calculate the directions of spectral energy transfer of triadic interactions in MHD turbulence depending on the helicities of the interacting modes and on the ratio of magnetic and kinetic energies [4].

HELICAL MODE INTERACTIONS

The Fourier components of the velocity and magnetic fields $\boldsymbol{u}(\boldsymbol{k})$ and $\boldsymbol{b}(\boldsymbol{k})$ can be decomposed into helical modes

$$\boldsymbol{u}(\boldsymbol{k},t) = a_-(\boldsymbol{k},t)\boldsymbol{h}_-(\boldsymbol{k}) + a_+(\boldsymbol{k},t)\boldsymbol{h}_+(\boldsymbol{k}) = \sum_{s_k} a_{s_k}(\boldsymbol{k},t)\boldsymbol{h}_{s_k}(\boldsymbol{k}) \ , \quad (1)$$

$$\boldsymbol{b}(\boldsymbol{k},t) = c_-(\boldsymbol{k},t)\boldsymbol{h}_-(\boldsymbol{k}) + c_+(\boldsymbol{k},t)\boldsymbol{h}_+(\boldsymbol{k}) = \sum_{s_k} c_{s_k}(\boldsymbol{k},t)\boldsymbol{h}_{s_k}(\boldsymbol{k}) \ , \quad (2)$$

where a_s and c_s are complex amplitudes and $\boldsymbol{h}_s$ eigenvectors of the curl operator. The subscript $s = \pm 1$ denotes positive or negative helicity. Substituting the helical decompositions of $\boldsymbol{u}$ and $\boldsymbol{b}$ into the Fourier-transformed MHD equations leads to the following system of coupled ordinary differential equations (ODEs) for the evolution of the interaction of a single triad of wavevectors $\boldsymbol{k}, \boldsymbol{p}$ and $\boldsymbol{q} = \boldsymbol{k} - \boldsymbol{p}$, once the dissipative terms have been neglected [3]

$$\begin{aligned}
\partial_t a_{s_k} &= (s_p p - s_q q)(-\frac{1}{4}\boldsymbol{h}^*_{s_p} \times \boldsymbol{h}^*_{s_q} \cdot \boldsymbol{h}^*_{s_k})(a^*_{s_p} a^*_{s_q} - c^*_{s_p} c^*_{s_q}) \ , \\
\partial_t a_{s_p} &= (s_q q - s_k k)(-\frac{1}{4}\boldsymbol{h}^*_{s_q} \times \boldsymbol{h}^*_{s_k} \cdot \boldsymbol{h}^*_{s_p})(a^*_{s_q} a^*_{s_k} - c^*_{s_q} c^*_{s_k}) \ , \\
\partial_t a_{s_q} &= (s_k k - s_p p)(-\frac{1}{4}\boldsymbol{h}^*_{s_k} \times \boldsymbol{h}^*_{s_p} \cdot \boldsymbol{h}^*_{s_q})(a^*_{s_k} a^*_{s_p} - c^*_{s_k} c^*_{s_p}) \ , \quad (3)
\end{aligned}$$

$$
\begin{aligned}
\partial_t c_{s_k} &= -s_k k(-\frac{1}{2}\boldsymbol{h}^*_{s_p} \times \boldsymbol{h}^*_{s_q} \cdot \boldsymbol{h}^*_{s_k})(a^*_{s_p} c^*_{s_q} - c^*_{s_p} a^*_{s_q}) \ , \\
\partial_t c_{s_p} &= -s_p p(-\frac{1}{2}\boldsymbol{h}^*_{s_q} \times \boldsymbol{h}^*_{s_k} \cdot \boldsymbol{h}^*_{s_p})(a^*_{s_q} c^*_{s_k} - c^*_{s_q} a^*_{s_k}) \ , \\
\partial_t c_{s_q} &= -s_q q(-\frac{1}{2}\boldsymbol{h}^*_{s_k} \times \boldsymbol{h}^*_{s_p} \cdot \boldsymbol{h}^*_{s_q})(a^*_{s_k} c^*_{s_p} - c^*_{s_k} a^*_{s_p}) \ . \qquad (4)
\end{aligned}
$$

DIRECTION OF SPECTRAL ENERGY TRANSFER PROCESSES

In order to derive the direction of energy transfer depending on the helicities of the interacting modes, we follow Waleffe's approach and study the stability of steady solutions to this system of ODEs.

Without loss of generality we impose an ordering of wavenumbers k, p and q as $k < p < q$, and to simplify the notation, we write the helical $\boldsymbol{u}$- and $\boldsymbol{b}$-field amplitudes as:

$$
(a_{s_k}, a_{s_p}, a_{s_q}) \text{ and } (c_{s_k}, c_{s_p}, c_{s_q}) \qquad (5)
$$

The non-trivial steady solutions of the system (3) - (4) are found to be

$$
(A_{s_k}, 0, 0) \text{ and } (C_{s_k}, 0, 0) \ , (0, A_{s_p}, 0) \text{ and } (0, C_{s_p}, 0) \ , (0, 0, A_{s_q}) \text{ and } (0, 0, C_{s_q}) \ ,
$$

and are of the same form as in [1] (following the notation in (5)), where two modes have zero energy and the nonzero energy one is constant in time. We now study the stability of the positive energy amplitudes with respect to infinitesimal perturbations of the zero-energy amplitudes. If the perturbations grow, then the positive energy mode is unstable and we assume that energy is transferred from the unstable mode to the two other modes it interacts with in a given k, p, q triad. This has been named *instability assumption* [1], inspired by the formal analogy to rigid-body rotation, where rotation around the axis of middle inertia is unstable.

In order to consider an example case, we assume the modes corresponding to the middle wavenumber to be constant in time. This leads to a matrix ODE for the perturbations at e.g. wavenumber $k < p$

$$
\partial_t^2 \boldsymbol{x}_{s_k}(t) = A(k, p, q, s_k, s_p, s_q, |A_{s_p}|, |C_{s_p}|)\boldsymbol{x}_{s_k}(t) \ , \qquad (6)
$$

where $\boldsymbol{x}_{s_k}(t) = (c_{s_k}(t), a_{s_k}(t))$ and A is a 2×2 matrix depending on the helicities of the interacting modes and the magnitudes of the nonzero energy modes A_p and C_p. The stability of solutions of the system (6) depends on the eigenvalues of A. If the real part of the square root of at least one eigenvalue is positive, $\boldsymbol{x}_{s_k}(t)$ will grow exponentially, that is the steady solution $(0, A_{s_p}, 0), (0, C_{s_p}, 0)$ is unstable and energy will be transferred away from the velocity and magnetic field modes at wavenumber p into velocity and magnetic field modes at wavenumbers k and q. There are two special cases in which the solutions are easy to obtain. These are

$$
(0, A_{s_p} = 0, 0) \ , \ (0, C_{s_p} \neq 0, 0) \text{ and } (0, A_{s_p} \neq 0, 0) \ , \ (0, C_{s_p} = 0, 0) \ .
$$

In the first case of this example we obtain energy transfer from C_{s_p} into c_{s_k} if $s_p = s_k$ and no transfer from C_{s_p} into a_{s_k} and c_{s_q}. In the second case we obtain energy transfer from A_{s_p} into c_{s_k} and c_{s_q} if $s_q \neq s_k$ and into a_{s_k} if $s_p = s_q$, which is the solution found by Waleffe for the Navier-Stokes case.

In the general case the direction of energy transfer depends not only on the helicities of the interacting modes but also on the magnitudes of the nonzero helical amplitudes A_{s_p} and C_{s_p}. For $|A_{s_p}| > |C_{s_p}|$ and $s_q = s_p \neq s_k$. energy is transferred from magnetic and velocity field modes at p into magnetic and velocity field modes at k. For $|C_{s_p}| > |A_{s_p}|$ and $s_q = s_p = s_k$ energy is transferred from magnetic and velocity field modes at p into magnetic and velocity field modes at k, provided $|p - q| < |p - k|$. This can happen for small wavenumbers k interacting with large wavenumbers p and q, as the triad geometry then forces p and q to be of similar magnitude.

In summary, one obtains from the example case above that interactions of modes with the same helicities facilitate reverse spectral transfer from larger to smaller wavenumbers.

References

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