BUILDING PROPER INVARIANTS FOR SUBGRID-SCALE EDDY-VISCOSITY MODELS

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Abstract Since direct simulations of the incompressible Navier-Stokes equations are limited to relatively low-Reynolds numbers, dynamically less complex mathematical formulations are necessary for coarse-grain simulations. Eddy-viscosity models for Large-Eddy Simulation is probably the most popular example thereof: they rely on differential operators that should be able to capture well different flow configurations (laminar and 2D flows, near-wall behavior, transitional regime...). Most of them are based on the combination of invariants of a symmetric second-order tensor that is derived from the gradient of the resolved velocity field. In the present work, they are presented in a framework where all the models are represented as a combination of elements of a 5D phase space of invariants. In this way, new models can be constructed by imposing appropriate restrictions in this space. The performance of the proposed models is successfully tested for a turbulent channel flow.

THEORY: A 5D PHASE SPACE FOR EDDY-VISCOSITY MODELS

Due to its inherent simplicity and robustness, the eddy-viscosity assumption is by far the most popular closure to model the subgrid-scales in Large-Eddy Simulation. In order to be frame invariant, they are usually based on the combination of invariants of a symmetric second-order tensor that depends on the gradient of the resolved velocity field, $G \equiv \nabla \overline{u}$. This second-order traceless tensor, $tr(G) = \nabla \cdot \overline{u} = 0$, contains 8 independent elements and can be characterized by 5 invariants (3 scalars are required to specify the orientation in 3D). Following the same notation as in [1], this set of five invariants can be defined as follows

$$\{Q_G, R_G, Q_S, R_S, V^2\}, \tag{1}$$

where $Q_A = 1/2\{tr^2(A) - tr(A^2)\}$ and $R_A = det(A)$ represent the second and third invariants of the second-order tensor A, respectively. Moreover, the first invariant of A is denoted as $P_A = tr(A)$. Finally, V^2 is equal to the L^2-norm of the vortex-stretching vector, *i.e.* $V^2 = 4(tr(S^2\Omega^2) - 2Q_S Q_\Omega) = |S\omega|^2 \geq 0$, where $S = 1/2(G + G^T)$, $\Omega = 1/2(G - G^T)$ and $\omega = \nabla \times u$. Starting from the classical Smagorinsky model [5] that reads

$$\nu_e^{Smag} = (C_S \Delta)^2 |S(\overline{u})| = 2(C_S\Delta)^2(-Q_S)^{1/2}, \tag{2}$$

existing models can be re-written in terms of the 5D phase space defined in (1). For instance, the WALE [3] and the Vreman's model [8] respectively read

$$\nu_e^W = (C_W \Delta)^2 \frac{(V^2/2 + 2/3Q_G^2)^{3/2}}{(-2Q_S)^{5/2} + (V^2/2 + 2/3Q_G^2)^{5/4}} \quad \text{and} \quad \nu_e^{Vr} = (C_{Vr}\Delta)^2 \left(\frac{V^2 + Q_G^2}{2(Q_\Omega - Q_S)} \right)^{1/2}, \tag{3}$$

where $Q_\Omega = Q_G - Q_S$. The major drawback of the Smagorinsky model is that the differential operator it is based on does not vanish in near-wall regions (see Figure 1, right). It is possible to build models based on invariants without this limitation. Examples thereof are the WALE, the Vreman's, the Verstappen's and the σ-model (see also Figure 1, right).

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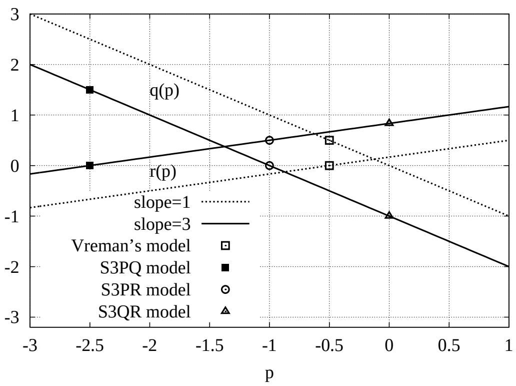
At this point, it is interesting to observe that new models can be derived by imposing restrictions on the differential operators they are based on. For instance, let us consider models that are based on the invariants of the tensor GG^T

$$\nu_e = (C_M\Delta)^2 P^p_{GG^T} Q^q_{GG^T} R^r_{GG^T}, \quad \text{where} \quad -6r - 4q - 2p = -1; \qquad 6r + 2q = s, \tag{4}$$

and $P_{GG^T} = 2(Q_\Omega - Q_S)$, $Q_{GG^T} = V^2 + Q_G^2$ and $R_{GG^T} = R_G^2$, respectively. The above-defined restrictions on the exponents follow by imposing the $[T^{-1}]$ units of the differential operator and the slope, s, for the asymptotic near-wall behavior (see Figure 1, right), *i.e.* $\mathcal{O}(y^s)$. Solutions for $q(p, s) = (1 - s)/2 - p$ and $r(p, s) = (2s - 1)/6 + p/3$ are displayed in Figure 1. The Vreman's model given in Eq.(3) corresponds to the solution with $s = 1$ (see Figure 1) and $r = 0$. However, it seems more appropriate to look for solutions with the proper near-wall behavior, *i.e.* $s = 3$ (solid lines in Figure 1). Restricting to solutions involving only two invariants, the three models (also represented in Figure 1) follow,

$$\nu_e^{S3QP} = (C_{s3qp}\Delta)^2 P^{-5/2}_{GG^T} Q^{3/2}_{GG^T}; \quad \nu_e^{S3RP} = (C_{s3rp}\Delta)^2 P^{-1}_{GG^T} R^{1/2}_{GG^T}; \quad \nu_e^{S3RQ} = (C_{s3rq}\Delta)^2 Q^{-1}_{GG^T} R^{5/6}_{GG^T}, \tag{5}$$



Invariants

Q_G	R_G	Q_S	R_S	V^2	Q_Ω
$\mathcal{O}(y^2)$	$\mathcal{O}(y^3)$	$\mathcal{O}(y^0)$	$\mathcal{O}(y^1)$	$\mathcal{O}(y^2)$	$\mathcal{O}(y^0)$
$[T^{-2}]$	$[T^{-3}]$	$[T^{-2}]$	$[T^{-3}]$	$[T^{-4}]$	$[T^{-2}]$

Models

Smagorinsky Eq.(2)	WALE Eq.(3)	Vreman's Eq.(3)	Verstappen's Ref. [7]	σ-model Ref. [4]
$\mathcal{O}(y^0)$	$\mathcal{O}(y^3)$	$\mathcal{O}(y^1)$	$\mathcal{O}(y^1)$	$\mathcal{O}(y^3)$

Figure 1. Left: Solutions for the linear system of Eqs.(4) for $s = 1$ (dashed line) and $s = 3$ (solid line). Each (r, q, p) solution represents an eddy-viscosity model of the form given in Eq.(4). Right: near-wall behavior and units of the five basic invariants in the 5D phase space given in (1) and the invariant $Q_\Omega = Q_G - Q_S$ together with the near-wall behavior of several eddy-viscosity models.

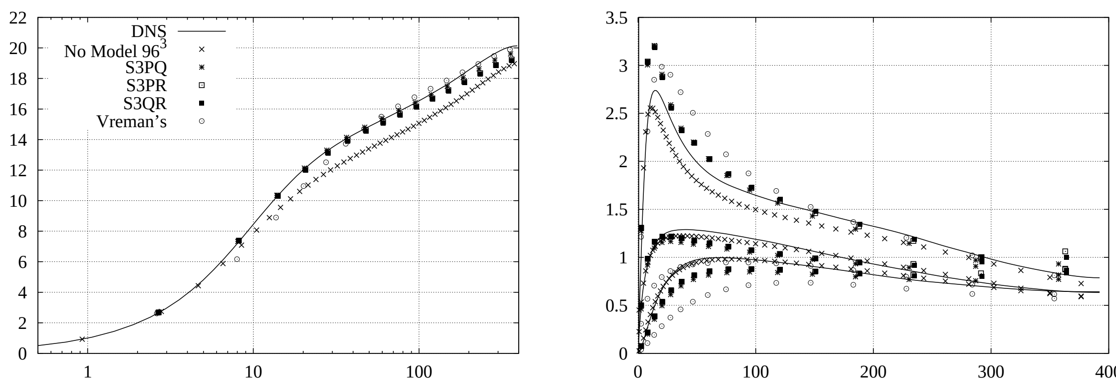


Figure 2. Results for a turbulent channel flow at $Re_\tau = 395$ obtained with a 32^3 mesh for LES and a 96^3 mesh without model, *i.e.* $\nu_e = 0$. Solid line corresponds to the DNS by Moser *et al.* [2].

where the model constants, C_{s3xx}, can be related with the Vreman's constant, C_{Vr}, with the following inequality

$$0 \leq \frac{(C_{Vr})^2}{(C_{s3xx})^2}\frac{\nu_e^{S3xx}}{\nu_e^{Vr}} \leq \frac{1}{3}. \tag{6}$$

Hence, imposing $C_{s3pq} = C_{s3pr} = C_{s3qr} = \sqrt{3}C_{Vr}$ guarantees both numerical stability and that the models have less or equal dissipation than Vreman's model, *i.e.* $0 \leq \nu_e^{S3xx} \leq \nu_e^{Vr}$. Figure 2 shows the performance of the proposed models for a turbulent channel flow in conjunction with the discretization methods for eddy-viscosity models proposed in [6]. Compared with Vreman's model, they improve the results near the wall.

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