GOING BEYOND EDDY VISCOSITY: FINDING A MINIMAL REPRESENTATION OF SUBGRID-SCALE STRESSES IN LARGE-EDDY SIMULATION

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Abstract In the current study we aim to go beyond the dissipative description of turbulent flows that is provided by eddy viscosity models for large-eddy simulation. As a starting point, we consider a general subgrid-scale model that is nonlinear in the velocity gradient. To reduce the number of degrees of freedom of the model, we propose a first-principles-based procedure to find a minimal representation of subgrid-scale stresses. Then, several criteria to determine the dependence of model coefficients on flow properties are detailed. Ultimately, this would lead to a better understanding of the role of different nonlinear model terms in the description of turbulent flows.

INTRODUCTION

We study the construction of subgrid-scale models for large-eddy simulation of incompressible turbulent flows. In large-eddy simulation one seeks to predict the behavior of the larger scales of motion within a flow field. Usually, the distinction between large and small scales is made by a filtering or coarse-graining operation, and the evolution of the large-scale velocity field is given by the filtered Navier-Stokes equations,

$$\frac{\partial \bar{u}_i}{\partial x_i} = 0, \qquad \frac{\partial \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j}(\bar{u}_i\bar{u}_j) = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \nu\frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j} - \frac{\partial}{\partial x_j}(\underbrace{\overline{u_i u_j} - \bar{u}_i\bar{u}_j}_{\tau_{ij}}). \tag{1}$$

Here, the subgrid-scale stress tensor, τ, represents the interactions between large and small scales of motion. As it is not solely expressed in terms of the large-scale, filtered velocity field it cannot be resolved in a numerical simulation and it has to be modeled. The subgrid-scale models we consider here depend on the filtered rate-of-strain and rate-of-rotation tensors,

$$\bar{S}_{ij} = \frac{1}{2}(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}), \qquad \bar{\Omega}_{ij} = \frac{1}{2}(\frac{\partial \bar{u}_i}{\partial x_j} - \frac{\partial \bar{u}_j}{\partial x_i}). \tag{2}$$

EXAMPLES OF SUBGRID-SCALE MODELS

Based on the idea that small-scale turbulent motions effectively cause diffusion of the larger scales, in eddy viscosity models the off-diagonal subgrid-scale stresses are often taken proportional to the rate of strain,

$$\tau^{model} - \tfrac{1}{3}\,\mathrm{Tr}(\tau^{model})I = -2\nu_e\bar{S}. \tag{3}$$

When the eddy viscosity, ν_e, is chosen properly, this model captures the net transfer of energy from large to small scales, sometimes referred to as the subgrid-scale dissipation. Unless ν_e is taken negative, however, the reverse process of backscatter cannot be captured. Furthermore, the model incorrectly imposes alignment of the eigenvectors of the subgrid-scale stresses with those of the rate-of-strain tensor [6].
A different subgrid-scale model, which can be constructed by approximating the filtering operation in the subgrid-scale stress tensor, is the gradient model. In terms of the filtered rate of strain and rate of rotation it is given by

$$\tau^{model} = C(\bar{S}^2 - \bar{\Omega}^2 - (\bar{S}\bar{\Omega} - \bar{\Omega}\bar{S})). \tag{4}$$

In several *a priori* studies it has been shown to capture the eigenvector orientations of the actual subgrid-scale stresses better [6]. Also, there are suggestions that it is this tensor structure, rather than a negative eddy viscosity, that is related to backscatter [2]. The gradient model has its deficiencies, however. It is inherently unstable as it does not transport enough energy to smaller scales. Usually this problem is remedied by taking a linear combination of the above models, resulting in a mixed model by which forward and backward scatter can be represented. [9]

A GENERAL NONLINEAR SUBGRID-SCALE MODEL

Motivated by the results provided by mixed models, we consider a general subgrid-scale model that is nonlinear in the velocity gradient. It is constructed by assuming that the subgrid-scale stress tensor can be expressed as a function of the filtered rate-of-strain and rate-of-rotation tensors, $\bar{S}$ and $\bar{\Omega}$. From the Cayley-Hamilton theorem it then follows that the model can be represented by the linear combination [3],

$$\tau^{model} = \alpha_i T_i, \tag{5}$$

of a finite number of tensors,

$$\begin{aligned} &T_0 = I, && T_3 = \bar{\Omega}^2, && T_7 = \bar{\Omega}\bar{S}\bar{\Omega}^2 - \bar{\Omega}^2\bar{S}\bar{\Omega}, \\ &T_1 = \bar{S}, && T_4 = \bar{S}\bar{\Omega} - \bar{\Omega}\bar{S}, && T_8 = \bar{S}\bar{\Omega}\bar{S}^2 - \bar{S}^2\bar{\Omega}\bar{S}, \\ &T_2 = \bar{S}^2, && T_5 = \bar{S}^2\bar{\Omega} - \bar{\Omega}\bar{S}^2, && T_9 = \bar{S}^2\bar{\Omega}^2 + \bar{\Omega}^2\bar{S}^2, \\ & && T_6 = \bar{S}\bar{\Omega}^2 + \bar{\Omega}^2\bar{S}, && T_{10} = \bar{\Omega}\bar{S}^2\bar{\Omega}^2 - \bar{\Omega}^2\bar{S}^2\bar{\Omega}. \end{aligned} \tag{6}$$

The coefficients, α_i , may depend on the following invariants of $\bar{S}$ and $\bar{\Omega}$,

$$\begin{aligned} &I_1 = \mathrm{Tr}(\bar{S}^2), && I_3 = \mathrm{Tr}(\bar{S}^3), && I_5 = \mathrm{Tr}(\bar{S}^2\bar{\Omega}^2), \\ &I_2 = \mathrm{Tr}(\bar{\Omega}^2), && I_4 = \mathrm{Tr}(\bar{S}\bar{\Omega}^2), && I_6 = \mathrm{Tr}(\bar{S}^2\bar{\Omega}^2\bar{S}\bar{\Omega}). \end{aligned} \tag{7}$$

FINDING A MINIMAL REPRESENTATION OF SUBGRID-SCALE STRESSES

The general model of Eqs. (5)–(7) is expected to allow for a better representation of subgrid-scale stresses than the eddy viscosity and gradient models, especially in wall-bounded and rotating flows. Containing as many as eleven adjustable constants, the model seems to be unnecessarily complicated, however. Indeed, as Lund and Novikov [3] remark, in most cases, only six out of the eleven tensors suffice to describe the degrees of freedom of the subgrid-scale stress tensor.

So far, in practical model tests, a smaller subset of the above model terms is used. For instance, Marstorp et al. [4] derive a model consisting of three basis tensors from the evolution equation of the deviatoric part of the subgrid-scale stress tensor. Wang and Bergstrom [8] take a different set of four terms. For an extensive review of the use of these and similar nonlinear models in the RANS community, see [1].

In the present work, rather than discarding any of the above tensors, we will extend the analysis of [1] and [3], and perform a Gram-Schmidt orthogonalization process to isolate all independent contributions, say T_i'. These are then used to form a minimal representation of the subgrid-scale stresses, that is, a model of the form $\alpha_i' T_i'$ consisting of the smallest set of tensors that contains the same number of degrees of freedom as τ. For this minimal representation, we look to determine the functional dependence of the model coefficients, α_i', on flow properties, based on analytical considerations.

The work by Vreman [7] shows how this can be done for the term linear in $\bar{S}$. He investigates the subgrid dissipation of the actual subgrid-scale stress tensor and demands that for all flows for which it is zero, also the model subgrid dissipation vanishes. In formula form,

$$\tau_{ij}\bar{S}_{ij} = 0 \rightarrow \tau_{ij}^{model}\bar{S}_{ij} = 0. \tag{8}$$

We propose to extend this first-principles analysis to the case of the general nonlinear model by requiring that the modeled subgrid-scale force vanishes for all flow locations at which there is no actual subgrid-scale force,

$$\frac{\partial}{\partial x_j}\tau_{ij} = 0 \rightarrow \frac{\partial}{\partial x_j}\tau_{ij}^{model} = 0. \tag{9}$$

Other analytical criteria to restrict the model coefficients are under study, such as the preservation of well-known symmetries of the Navier-Stokes equations [5]. Also, *a priori* model tests are planned to determine the magnitude of the different model terms in canonical turbulent flows.

Ultimately, the construction of a nonlinear model that is based on a minimal representation of subgrid-scale stresses and the knowledge of the dependence of its coefficients on flow properties would lead to a better understanding of the physics represented by each of the model terms and of their role in the description of turbulent flows.

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