SCALING PROPERTIES OF COMPRESSIBLE POLYTROPIC TURBULENCE

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Abstract An exact relation has been derived for homogeneous polytropic turbulence in terms of the two-point fluctuations. Unlike isothermal turbulence, the fluctuating sound speed appears to be a key factor in determining the scaling properties of polytropic turbulence. Three Mach numbers (current, gradient and turbulent Mach number) are defined to characterize the isotropic scaling. At subsonic scale, the Kolmogorov-like -5/3 energy spectrum for $\rho^{1/3}\mathbf{v}$ (obtained for isothermal turbulence) is modified by polytropic contribution. For supersonic turbulence, the source terms must be taken into account which further complicates the scaling and spectral properties. For highly supersonic case, the source terms, however, can be shown to be nearly equal to the source terms in isothermal turbulence thereby leading to a partial simplification. These theoretical results are extremely important for understanding the role of turbulence in the star-formation mechanism in polytropic clouds.

INTRODUCTION

Proper understanding of supersonic turbulence is essential for explaining different aspects of interstellar turbulence including the fundamental question of star formation in turbulent interstellar clouds [3, 10]. Despite considerable advancements in the theory of incompressible turbulence, it is only recently that important efforts [8, 9, 5] have been made in order to understand the nature of three dimensional compressible turbulence. Following Kolmogorov [7], we derived two exact relations for isothermal hydrodynamic [6] and magnetohydrodynamic turbulence [1] and could explain the universality of isothermal turbulence in terms of cube-root density weighted dynamic variables (velocity, Elsässer variables etc.). Real interstellar (molecular) clouds, however, undergo considerable temperature fluctuations which can be approximated by a polytropic closure of type

$$P = K\rho^{\gamma},$$

where P and ρ are respectively the fluid pressure and density , K being the proportionality constant and γ is the polytropic index. In practical cases, γ of an interstellar cloud varies between 0.5 (very dilute interstellar media with density ~ 10 protons/c.c) and 1.67 (dense cores with density $\sim 10^{21}$ protons/c.c and appropriate for star formation). In a recent work [4], the dependence of star forming rate (SFR) of a cloud and its polytropic index is studied thoroughly and a variation of factor 5 in SFR is reported within the aforesaid range of variation of γ. An analytical understanding of polytropic turbulence is therefore crucial for probing into the corresponding physics.

EXACT RELATION FOR POLYTROPIC TURBULENCE

In course of this presentation, I would like to present our recent theoretical work [2] on compressible polytropic turbulence and to discuss analytically derived scaling properties (both in physical space and in Fourier space) of such turbulence. Using the hypothesis of statistical homogeneity, in the case of non-isotropic polytropic turbulence, we obtain,

$$-2\varepsilon = \nabla_{\mathbf{r}} \cdot \left\langle \frac{1}{2} \left(\delta\left(\rho \mathbf{v}\right) \cdot \delta \mathbf{v}\right) \delta \mathbf{v} + \frac{1}{\gamma(\gamma-1)} \delta\left(\rho C_s\right) \delta C_s \delta \mathbf{v} + \overline{\delta} h \delta(\rho \mathbf{v}) \right\rangle + \mathcal{S}, \qquad (1)$$

where $\mathcal{S} = \left\langle (\nabla\cdot\mathbf{v}) \left(R'_E - E' + \frac{P'}{2} - \frac{1}{\gamma}\overline{\delta}\rho\, C_s C'_s\right) + (\nabla'\cdot\mathbf{v}') \left(R_E - E + \frac{P}{2} - \frac{1}{\gamma}\overline{\delta}\rho\, C_s C'_s\right)\right\rangle$ represents the source term, δX gives two point-fluctuation of a variable X, $\overline{\delta} X \equiv (X + X')/2$ stands for two-point average of any variable X, $\mathbf{v}$ is the fluid velocity, C_S is polytropic sound speed, h is enthalpy per unit mass, E is the total energy (kinetic + compressive energy), R_E is two-point energy correlator and ε represents the total energy flux rate.
Unlike isothermal case, for a polytropic fluid, sound speed is no longer a constant. The competition between fluctuation of fluid velocity and the fluctuation of sound speed is believed to play a key role in polytropic turbulence (as it is evident from the flux term of equation (1). Moreover, the source terms are more complicated than the isothermal hydrodynamic case due to the sound speed fluctuations.

ISOTROPIC TURBULENCE AND PLAUSIBLE ENERGY SPECTRA

Under statistical isotropy, the exact relation can be dimensionally written as

$$-2\varepsilon_\ell \simeq \frac{(\rho v)_\ell\, v_\ell^2}{\ell} \left(\frac{1}{2} + \frac{1}{\gamma(\gamma-1) M_{\rho_\ell} M_\ell} + \frac{1}{(\gamma-1)\,\mathcal{M}_\ell^2} + \frac{1}{4\,(\gamma-1)\,M_\ell^2} \right) + \mathcal{S}_\ell\,, \qquad (2)$$

where:

$$M_{\rho\ell} \equiv \frac{\delta(\rho v)}{\delta(\rho C_s)} \sim \frac{(\rho v)_\ell}{(\rho C_s)_\ell}, \quad M_\ell \equiv \frac{\delta v}{\delta C_s} \sim \frac{v_\ell}{C_{s\ell}}, \quad \mathcal{M}_\ell \equiv \frac{\delta v}{\bar{\delta} C_s} \sim \frac{v_\ell}{\bar{C}_s}, \tag{3}$$

are respectively the *current* Mach number, the *gradient* Mach number (which is not defined for isothermal turbulence where the sound speed is constant) and the *turbulent* Mach number and ℓ is the fluctuation length scale. The third one is familiar in turbulence studies whereas the first and the second one have been defined for the sake of our current study. The introduction of the *gradient* Mach number refines the sub-sonic regime into two sub-regimes depending whether $|\delta \mathbf{v}| < \delta C_S$ or $|\delta \mathbf{v}| > \delta C_S$ thereby introducing an important difference with respect to isothermal turbulence.

Under the assumption that the density has a different (usually much smaller) correlation length than that of the fluid velocity and sound speed, the current Mach number can be approximated as gradient Mach number and the power spectra for density-weighted velocity $\mathbf{w} = \rho^{1/3}\mathbf{v}$ can be written as

$$E_k^w \sim \varepsilon_{eff}^{\frac{2}{3}} k^{-\frac{5}{3}} \left(1 + \Gamma_1 k^{2\alpha} + \Gamma_2 k^{2\beta}\right)^{-\frac{2}{3}} , \tag{4}$$

where we assume two power-law scalings for the gradient and turbulent Mach numbers with respective indices α and β. ε_{eff} represents the net energy transfer rate through the flux term (and not the source terms) and $\Gamma_1 = (\gamma + 4)/[2\gamma(\gamma - 1)]$ and $\Gamma_2 = 2/(\gamma - 1)$ are explicit function of polytropic indices. In the above expression k represents isotropic wave number. Using the above expression (4), it is thus possible to predict an approximate power spectra for a turbulent polytropic fluid whose polytropic index and the Mach number scaling laws are known.

In order to obtain the complete spectral form, however, we have to know the scale dependence of the source terms (to know $\varepsilon_{eff}(\ell)$). For supersonic turbulence, the polytropic source term should behave like the isothermal case. It is therefore concluded that for the source, the polytropic behaviour is pronounced for sub-sonic case. An explicit theoretical prediction of the source term is however beyond the scope of our analytical approach. A numerical analysis is necessary to verify these theoretical predictions (work in progress with high resolution numerical simulations of non-isothermal simulations).

CONCLUSION

This work, to our knowledge, is the first analytical approach to understand polytropic turbulence. The sound speed fluctuations and the variations of polytropic indices distinguish the current case from the isothermal one. Different scaling laws and power density spectra are predicted assuming the universal behaviour of polytropic turbulence. Numerical experiments of polytropic turbulence are necessary to verify these theoretical results and also to better understand the phenomenology of interstellar turbulence.

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